Asian Journal of Probability and Statistics

12(1): 34-55, 2021; Article no.AJPAS.67001

ISSN: 2582-0230

Generalized Exponential Power Distribution with Application to Complete and Censored Data

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Authors' contributions

This work was carried out in collaboration between both authors. Author AEM defined the generalized exponential power distribution, mainly contributed Sects. 2 and 3 and revised the manuscript. Author MEA wrote the first draft of the manuscript and fully contributed Secs 4, 5, and 6. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2021/v12i130278

Editor(s):

(1) Dr. Dariusz Jacek Jakóbczak, Koszalin University of Technology, Poland.

Reviewers:

(1) Mohamed I. Marie, Helwan University, Egypt.

(2) Amirhossein Fathinavid, Islamic Azad University, Iran.

Complete Peer review History: http://www.sdiarticle4.com/review-history/67001

Received: 20 January 2021

Accepted: 25 March 2021

Published: 11 April 2021

Original Research Article

Abstract

The exponential power distribution (EP) is a lifetime model that can exhibit increasing and bathtub hazard rate function. This paper proposed a generalization of EP distribution, named generalized exponential power (GEP) distribution. Some properties of GEP distribution will be investigated. Recurrence relations for single moments of generalized ordered statistics from GEP distribution are established and used for characterizing the GEP distribution. Estimation of the model parameters are derived using maximum likelihood method based on complete sample, type I, type II and random censored samples. A simulation study is performed in order to examine the accuracy of the maximum likelihood estimators of the model parameters. Three applications to real data, two with censored data, are provided in order to show the superiority of the proposed model to other models.

Keywords: Exponential power distribution; lifetime data; hazard function; generalized exponential power distribution; generalized order statistics; censored data.

2010 Mathematics Subject Classification: 60E05, 62E10, 62E15.

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1 Introduction

The exponential power (EP) distribution with bathtub shape or increasing hazard rate is proposed by Smith and Bain [1]. Its distribution function is given by

$$G_{EP}(x) = 1 - e^{-\left(e^{\lambda x^{\alpha}}-1\right)}, \tag{1.1}$$

where $\alpha > 0$ is a shape parameter and $\lambda > 0$ is a scale parameter. This distribution may be thought of as a truncated extreme-value distribution with a Weibull type parameterization rather than the usual location-scale parameterization.

An extension of EP distribution has been proposed by Barriga et al. [2] based on family of distributions given by Lehman alternatives (called exponentiated type family by Nadarajah and Kotz [3]) considered by Gupta et al. [4]. Based on modification of the EP distribution, Chen [5] proposed two parameter distribution with bathtub or increasing hazard rate function. Xei et al. [6] proposed an extension of Chens model, known as the Weibull extension model, by adding scale parameter. It has been further extended by Pappas et al. [7] on the latent of competing risk problems, Basu and Klein [8], using the technique of Marshall and Olkin [9]. Another extension proposed by Chaubey and Zhang [10] according to Lehman alternatives, called the extended Chen (EC) family of distributions. Other generalizations in the literature can be found in [11, 12, 13, 14], etc.

Complementary risks problem, [8], is a key concept arises in survival analysis. Simplistically, we only observe the maximum component lifetime of a parallel system, which is the cause of failure for the system. Another key concept is frailty models which arises in survival analysis to assess the possible heterogeneity that may appears between the population individuals. In this case, the population individuals don't have the same risk to fail (they have different frail). For more details on frailty models, see [15, 16, 17]. This paper aims to propose a new four-parameter distribution called generalized exponential power (GEP) distribution. It is generated based on complementary risks problem and frailty model. In this way we can model the possible heterogeneity in a data set. So that, the proposed model can show a desirable flexibility in fitting real data sets.

The rest of this paper is organized as follows: Section 2 proposed the GEP distribution. Some of its statistical and reliability properties are provided in section 3. In Section 4, the estimation of the model parameters based on complete, type I, type II and random censored samples, using the maximum likelihood method, is discussed. A simulation study is performed, in section 5, to examine the accuracy of the maximum likelihood estimators of the model parameters. Finally, applications to three distinctive real data sets are presented in Section 6.

2 The GEP Distribution

Consider $X_i, i = 1, \ldots, N$, are independent and identically distributed random variables represent lifetimes of N components connected in parallel where N is geometric random variable, independent of X_s, with probability mass function given by $P(X = x) = p(1-p)^{n-1}, n = 1, 2, \ldots$. Let $F_0(x)$ be the baseline distribution function of X_s and $H_0(x)$ be the corresponding baseline cumulative hazard rate function. The system failure is due to the failure of the maximum component lifetime in such case the system undergoes a maintenance process. Thus, the heterogeneity may appears between its components. The notion of frailty models is introduced to assess this heterogeneity in a nice way. The classical and mostly applied frailty model assumes a proportional hazards model in which the hazard rate function of the i_{th} component depends additionally on an unobservable, age-independent, random variable Z_i which acts multiplicatively on the baseline hazard function

$$h_i(x|Z_i = z) = zh_0(x)$$

Let $Z_i, i = 1, \ldots, N$, are interpreted as independent and identically distributed non-negative random variables with common frailty distribution function $G(z)$. Thus, the survival function for the system components given the frailty Z_i is given by

$$\bar{F}_s(x|z) = e^{-zH_0(x)}$$

The mixture survival function for the system components is characterized by the laplace transform of the frailty distribution as follows

$$\bar{F}_s(x) = \int_0^\infty \bar{F}_s(x|z)dG(z) = L_G(H_0(x)) \tag{2.1}$$

Since, we only observe the lifetime of the last component to fail, $V = X_{(n)} = max\{X_1, X_2, \ldots, X_N\}$, the conditional distribution function of V given $N = n$ is given by

$$P(V \le x|N = n) = F_s(x)^n = (1 - L_G(H_0(x)))^n$$

Thus, the distribution function of the new family of distributions is the marginal distribution function of the last order statistic $V = X_{(n)}$ which given by

$$F(x) = \sum_{n=1}^{\infty} (1 - L_G(H_0(x)))^n p(1-p)^{n-1}.$$

$$F(x) = 1 - \frac{L_G(H_0(x))}{p + (1-p)\, L_G(H_0(x))}, \quad x > 0 \tag{2.2}$$

The pdf corresponding to the distribution function in (2.2) can be expressed by

$$f(x) = \frac{-ph_0(x)L_G^{'}(H_0(x))}{[p + (1-p)L_G(H_0(x))]^2} \tag{2.3}$$

The hazard rate function of the new family of distribution becomes

$$h(x) = \frac{-ph_0(x)L_G^{'}(H_0(x))}{L_G(H_0(x))\,[p + (1-p)L_G(H_0(x))]} \tag{2.4}$$

Consider the exponential power distribution with distribution function (1.1) as the baseline distribution for the system components with baseline cumulative hazard function $H_0(x) = Exp(\lambda x^\alpha - 1)$. The positive stable distribution is a frailty distribution displays heavy tail behavior, which makes it a good candidate for Zs. Its laplace transforms is given by $L(s) = Exp(-s^\beta)$. The EP distribution can be extended using Eq. (2.2), and (2.3). So that, the distribution function, survival function, density function and hazard rate function of the generalized exponential power (GEP) distribution are respectively given by

$$F(x) = 1 - \frac{e^{-\left(e^{\lambda x^\alpha}-1\right)^\beta}}{p + (1-p)\, e^{-\left(e^{\lambda x^\alpha}-1\right)^\beta}}, \tag{2.5}$$

$$\bar{F}(x) = \frac{e^{-\left(e^{\lambda x^\alpha}-1\right)^\beta}}{p + (1-p)\, e^{-\left(e^{\lambda x^\alpha}-1\right)^\beta}}, \tag{2.6}$$

$$f(x) = \frac{p\alpha\lambda\beta x^{\alpha-1} e^{\lambda x^\alpha} \left(e^{\lambda x^\alpha} - 1\right)^{\beta-1} e^{-\left(e^{\lambda x^\alpha}-1\right)^\beta}}{\left[p + (1-p)\, e^{-\left(e^{\lambda x^\alpha}-1\right)^\beta}\right]^2}, \tag{2.7}$$

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{p\alpha\lambda\beta x^{\alpha-1} e^{\lambda x^\alpha} \left(e^{\lambda x^\alpha} - 1\right)^{\beta-1}}{p + (1-p)\, e^{-\left(e^{\lambda x^\alpha}-1\right)^\beta}}, x > 0 \tag{2.8}$$

where α, λ, β and $p > 0$. The parameter λ controls the scale of GEP distribution, while p, α and β are shape parameters. The GEP distribution, with parameters α, λ, β and p, is reduced to EP distribution when $\beta = 1$ and $p = 1$.

Fig. 1. shows the possible shapes of the density function (2.7) of the GEP distribution.

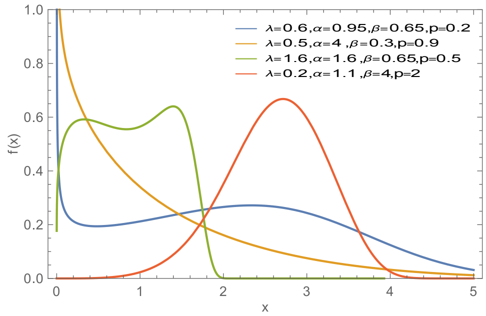


Fig. 1. The pdf of GEP distribution, for different values of parameters α, λ, β and p

The hazard rate function (2.8) is represented in fig. 2. for various values of α, λ, β and p. Fig. 2. shows a desirable flexibility of the hazard function of GEP distribution. It can accommodate increasing, bathtub, decreasing, unimodal and increasing-decreasing-increasing (IDI) hazard functions.

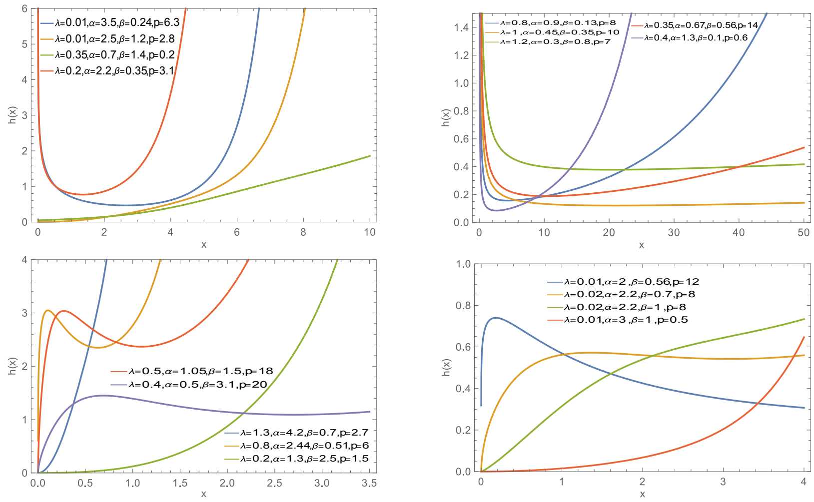


Fig. 2. Hazard function shapes of GEP distribution for different parameter values

An GEP distributed variable can be simulated using the inverse of the distribution function (2.5)

$$Q(u) = F_{GEP}^{-1}(u) = \frac{1}{\lambda^{\frac{1}{\alpha}}} \left[\ln \left(\left(\ln \left(\frac{1-(1-u)(1-p)}{p(1-u)} \right) \right)^{\frac{1}{\beta}} + 1 \right) \right]^{\frac{1}{\alpha}}, \tag{2.9}$$

where u has a uniform $U(0,1)$ distribution.

3 Properties

This section investigates some statistical and reliability properties of the GEP distribution.

3.1 Skewness and kurtosis

To study the effect of the shape parameters β and p of the GEP distribution on the skewness and kurtosis of the distribution, we plot the behavior of Galton skewness [18] and Moors kurtosis [19].

Galton skewness is

$$sk = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{2}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

Moors kurtosis is

$$ku = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}$$

where $Q(q)$ is the 100qth quantile of the GEP distribution, given by

$$Q\left(q\right) = \frac{1}{\lambda^{\frac{1}{\alpha}}}\left[\ln\left(\left(\ln\left(\frac{1-\left(1-q\right)\left(1-p\right)}{p\left(1-q\right)}\right)\right)^{\frac{1}{\beta}} + 1\right)\right]^{\frac{1}{\alpha}}, \; q \in \left(0,1\right). \tag{3.1}$$

Fig. 3. shows Galton skewness and moors kurtosis of GEP distribution for selected values of the shape parameters α, β and p with fixed scale parameter $\lambda = 1$.

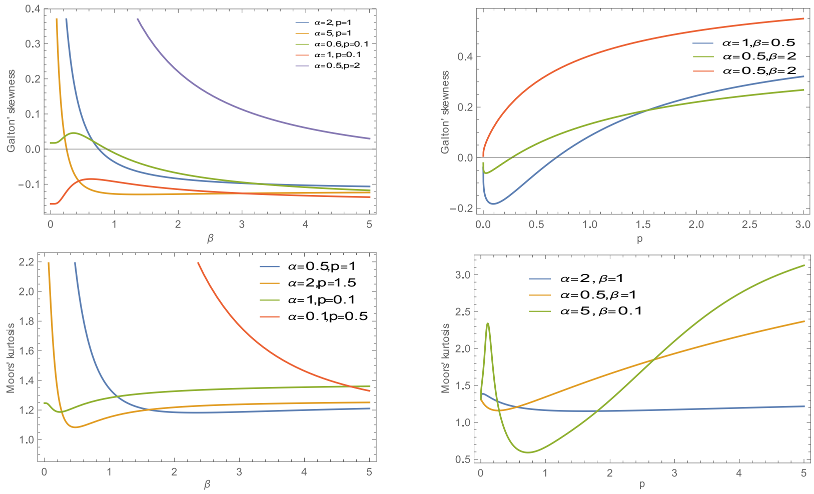


Fig. 3. Plots of Galton skewness and moors kurtosis of GEP distribution as a function of β and p

3.2 Moments

Many interesting characteristics and features of a distribution can be studied through its moments and incomplete moments. Let X be a random variable following the GEP distribution. The r_{th} ordinary moments of the random variable X, denoted by $\mu_r^{'}$, is the expected value of X^r

$$\mu_r^{'} = E\left(X^r\right) = \int_0^\infty x^r f(x)dx.$$

The density function of GEP distribution in (2.7) can be represented in series form as follow: the infinite series representation of $\left[p+(1-p)\,e^{-\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}\right]^{-2}$ or equivalently $p^{-2}\left[1-\left(\frac{p-1}{p}\right)e^{-\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}\right]^{-2}$ can be obtained, since $0<\left|\left(\frac{p-1}{p}\right)e^{-\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}\right|<1$ for $p>\frac{1}{2}$ and $\alpha,\beta>0$, as

$$f\left(x\right)=\alpha\lambda\beta\sum_{i=0}^{\infty}\frac{i+1}{p}\left(\frac{p-1}{p}\right)^{i}x^{\alpha-1}e^{\lambda x^{\alpha}}\left(e^{\lambda x^{\alpha}}-1\right)^{\beta-1}e^{-(i+1)\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}.$$

Now, using binomial expansion of $\left(e^{\lambda x^{\alpha}}-1\right)^{\beta-1}$, thus

$$f\left(x\right)=\alpha\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{(j)}\binom{\beta-1}{j}\frac{i+1}{p}\left(\frac{p-1}{p}\right)^{i}x^{\alpha-1}e^{(\beta-j)\lambda x^{\alpha}}e^{-(i+1)\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}. \tag{3.2}$$

Using series representation (3.2) the r_{th} ordinary moments of an GEP distributed random variable can be expressed as

$$\mu_{r}^{'}=\alpha\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{(j)}\binom{\beta-1}{j}\frac{i+1}{p}\left(\frac{p-1}{p}\right)^{i}\int_{0}^{\infty}x^{r+\alpha-1}e^{(\beta-j)\lambda x^{\alpha}}e^{-(i+1)\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}dx, \tag{3.3}$$

where $\alpha,\lambda,\beta>0$,$p>\frac{1}{2}$ and the integral $\int_{0}^{\infty}x^{r+\alpha-1}e^{(\beta-j)\lambda x^{\alpha}}e^{-(i+1)\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}dx$ can be obtained numerically via mathematical packages.

As a special case when $\beta=1$, $\mu_{r}^{'}$ can be written as

$$\mu_{r}^{'}=\alpha\lambda\sum_{i=0}^{\infty}\frac{i+1}{p}\left(\frac{p-1}{p}\right)^{i}\int_{0}^{\infty}x^{r+\alpha-1}e^{\lambda x^{\alpha}}e^{-(i+1)\left(e^{\lambda x^{\alpha}}-1\right)}dx,$$

Letting $y=e^{\lambda x^{\alpha}}$ in the above integral we get

$$\begin{aligned}\mu_{r}^{'}=&(\lambda)^{-\frac{r}{\alpha}}\sum_{i=0}^{\infty}e^{i+1}\frac{i+1}{p}\left(\frac{p-1}{p}\right)^{i}\int_{1}^{\infty}(ln\ y)^{\frac{r}{\alpha}}e^{-(i+1)y}dy\\ =&(\lambda)^{-\frac{r}{\alpha}}\sum_{i=0}^{\infty}e^{i+1}\frac{i+1}{p}\left(\frac{p-1}{p}\right)^{i}(\frac{r}{\alpha})!\mathbb{E}_{0}^{\frac{r}{\alpha}}(i+1),\end{aligned}$$

where $\mathbb{E}_{s}^{j}(z)=\frac{1}{(\frac{r}{\alpha})!}\int_{1}^{\infty}(ln\ y)^{j}y^{-s}e^{-zy}dy, \Re(j)>-1; s,z\in\mathbb{C}$, is an extension of generalized integro-exponential function introduced by Pogany et al. [20]. The function $\mathbb{E}_{s}^{j}(z)$ can be presented as

$$\mathbb{E}_{s}^{j}(z)=\sum_{l\geq 0}\frac{(s+2)_{l}}{l!}\Phi_{\mu,1}^{(0,1)}(-l,\jmath+1,1)\quad {}_{1}F_{1}(s+l+2;s+2;-z),$$

where $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,u)=\sum_{n\geq 0}\frac{(\mu)_{\rho n}z^{n}}{(\nu)_{\sigma n}(n+u)^{s}}$ is the Lin-Srivastava generalized Hurwitz-Lerch Zeta function, [21]. Here $(s+2)_{l}=\frac{\Gamma(s+l+2)}{\Gamma(s+2)}$ denotes the generalized Pochhammer symbol and ${}_{1}F_{1}(a;b;x)=\sum_{n\geq 0}\frac{(a)_{n}x^{n}}{(b)_{n}n!}$ is the confluent hypergeometric function - Kummer's function [22].

3.3 Incomplete moments and some related measures

The r_{th} incomplete moments of an GEP distributed random variable X is given by

$$m_r(t) = \int_0^t x^r f(x) dx.$$
$$= \alpha\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (-1)^{(j)} \binom{\beta-1}{j} \frac{i+1}{p} \left(\frac{p-1}{p}\right)^i \int_0^t x^{r+\alpha-1} e^{(\beta-j)\lambda x^\alpha} e^{-(i+1)\left(e^{\lambda x^\alpha}-1\right)^\beta} dx,$$

where $\alpha, \lambda, \beta > 0$,$p > \frac{1}{2}$ and the integral $\int_0^t x^{r+\alpha-1} e^{(\beta-j)\lambda x^\alpha} e^{-(i+1)\left(e^{\lambda x^\alpha}-1\right)^\beta} dx$ can be obtained numerically via mathematical packages.

As a special case when $\beta = 1$, $m_r(t)$ can be written as

$$m_r(t) = \alpha\lambda \sum_{i=0}^{\infty} \frac{i+1}{p} \left(\frac{p-1}{p}\right)^i \int_0^t x^{r+\alpha-1} e^{\lambda x^\alpha} e^{-(i+1)\left(e^{\lambda x^\alpha}-1\right)} dx,$$
$$= (\lambda)^{-\frac{r}{\alpha}} \sum_{i=0}^{\infty} e^{i+1} \frac{i+1}{p} \left(\frac{p-1}{p}\right)^i \int_1^{e^{\lambda t^\alpha}} (ln\ y)^{\frac{r}{\alpha}} e^{-(i+1)y} dy$$
$$= (\lambda)^{-\frac{r}{\alpha}} \sum_{i=0}^{\infty} e^{i+1} \frac{i+1}{p} \left(\frac{p-1}{p}\right)^i H(e^{\lambda t^\alpha}; i+1, \frac{r}{\alpha}),$$

where $H(q; z, j)$ is given by, Pogany et al. [20],

$$H(q; z, j) = \sum_{n,k\geq 0} \sum_{m=0}^{k} \frac{(2)_{n+k}}{(2)_n} \frac{(-1)^{m+n} z^n}{n!k!(m+1)^{j+1}} \binom{k}{m} \gamma(j, (1-q^{-1})(m+1))$$

where $\gamma(.,.)$ is lower incomplete gamma function.

Some important statistical measures are defined based on the moments and the incomplete moments, such as the mean deviation about the mean $D(\mu)$ and about the median D(M). These measures can be expressed as $D(\mu) = 2\mu F(\mu) - 2m_1(\mu)$ and $D(M) = \mu - 2m_1(M)$, where $\mu = E(X) = \mu_1^{'}$ and $M = Median(X) = Q(0.5)$. Table 1. provides small numerical study for the first four moments, variance, skewness, kurtosis and the mean deviations of GEP distribution for different scenarios of β and p with fixed $\lambda = 1$ and $\alpha = 1.5$

Another related measure is the mean residual life (MRL). The MRL is defined as the expected value of the remaining lifetimes for a unit after a fixed time point t. It can be defined in terms of the moments and incomplete moments as $mrl(t) = [\mu - m_1(t)]/[1 - F(t)] - t$. The MRL is related to the hazard function by the expression $h(t) = \frac{1+mrl^{'}(t)}{mrl(t)}$. It can be shown that for increasing (decreasing) hazard function the MRL is decreasing (increasing). Also, the mean inactivity time, which represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$ is given by $mit(t) = t - m_1(t)/F(t)$. Fig. 4. shows plots of MRL and MIT functions of GEP distribution for different parameters values.

An important application of the moments and incomplete moments is related to Bonferroni and Lorenz curves of X, which can be defined by $B(\pi) = m_1(q)/(\pi\mu)$ and $L(\pi) = m_1(q)/\mu$, respectively, where $q = Q(\pi)$ follows from (2.9) for a given probability π. The importance of Bonferroni and Lorenz curves is due to the wide variety of the potential applications of these curves. These curves can be applied in financial studies, medicine, and insurance. Plots for Bonferroni and Lorenz curves are presented in fig. 5.

Table 1. The first four moments, variance, skewness, kurtosis and mean deviations for different scenarios of β and p with fixed $\lambda = 1$ and $\alpha = 1.5$.

β	p	μ_1'	μ_2'	μ_3'	μ_4'	Variance	Skewness	Kurtosis	$D(\mu)$	$D(M)$
	0.2	1.9307	5.3482	16.4826	54.2391	1.6204	-0.0488	1.8572	1.0979	1.0935
	0.5	1.2760	3.0943	8.7898	27.3000	1.4661	0.6197	2.1922	1.0536	1.0354
	1	0.8483	1.8699	5.0509	15.1815	1.1504	1.2263	3.4483	0.8871	0.7953
0.2	1.5	0.6433	1.3493	3.5603	10.5491	0.9354	1.6456	4.8279	0.7555	0.6249
	2	0.5205	1.0576	2.7525	8.0896	0.7866	1.9824	6.2345	0.6570	0.5123
	3	0.3786	0.7397	1.8953	5.5208	0.5964	2.5267	9.0678	0.5218	0.3760
	4	0.2983	0.5693	1.4459	4.1914	0.4803	2.9731	11.9063	0.4337	0.2972
	0.2	1.2128	1.8320	3.0537	5.4272	0.3611	-0.2033	2.2300	0.4999	0.4977
	0.5	0.8955	1.1583	1.7444	2.8824	0.3563	0.3243	2.1886	0.5056	0.5048
	1	0.6680	0.7583	1.0593	1.6670	0.3120	0.7796	2.7670	0.4662	0.4563
0.5	1.5	0.5477	0.5740	0.7696	1.1817	0.2741	1.0803	3.4477	0.4271	0.4082
	2	0.4698	0.4650	0.6067	0.9174	0.2443	1.3142	4.1394	0.3940	0.3683
	3	0.3721	0.3397	0.4280	0.6353	0.2013	1.6807	5.5063	0.3427	0.3089
	4	0.3117	0.2689	0.3313	0.4864	0.1718	1.9726	6.8424	0.3049	0.2671
	0.2	0.9734	1.0520	1.2092	1.4527	0.1045	-0.5408	2.9226	0.2589	0.2565
	0.5	0.8001	0.7562	0.7857	0.8717	0.1159	-0.1236	2.3894	0.2803	0.2799
	1	0.6648	0.5572	0.5319	0.5545	0.1151	0.2152	2.3663	0.2814	0.2812
1	1.5	0.5871	0.4551	0.4124	0.4146	0.1104	0.4255	2.5244	0.2747	0.2736
	2	0.5335	0.3899	0.3403	0.3335	0.1053	0.5817	2.7252	0.2668	0.2645
	3	0.4614	0.3090	0.2555	0.2419	0.0961	0.8137	3.1535	0.2516	0.2471
	4	0.4131	0.2592	0.2062	0.1907	0.0885	0.9884	3.5784	0.2383	0.2321
	0.2	0.9013	0.8619	0.8574	0.8789	0.0496	-0.7897	3.6163	0.1744	0.1720
	0.5	0.7818	0.6700	0.6099	0.5806	0.0588	-0.4115	2.7909	0.1957	0.1947
	1	0.6846	0.5312	0.4474	0.4003	0.0625	-0.1146	2.5007	0.2044	0.2043
1.5	1.5	0.6267	0.4555	0.3651	0.3143	0.0628	0.0633	2.4649	0.2055	0.2055
	2	0.5856	0.4050	0.3129	0.2619	0.0621	0.1919	2.5013	0.2042	0.2041
	3	0.5286	0.3393	0.2481	0.1995	0.0598	0.3770	2.6442	0.1997	0.1990
	4	0.4891	0.2967	0.2084	0.1628	0.0575	0.5117	2.8151	0.1946	0.1934
	0.2	0.8680	0.7823	0.7246	0.6857	0.0290	-0.9632	4.2278	0.1314	0.1292
	0.5	0.7768	0.6392	0.5481	0.4851	0.0358	-0.6017	3.2044	0.1506	0.1493
	1	0.7009	0.5309	0.4250	0.3551	0.0395	-0.3240	2.7599	0.1608	0.1603
2	1.5	0.6548	0.4695	0.3595	0.2897	0.0408	-0.1612	2.6243	0.1641	0.1640
	2	0.6216	0.4276	0.3166	0.2483	0.0412	-0.0454	2.5820	0.1651	0.1651
	3	0.5748	0.3714	0.2616	0.1971	0.0410	0.1180	2.5975	0.1647	0.1646
	4	0.5418	0.3339	0.2265	0.1657	0.0404	0.2344	2.6619	0.1630	0.1627
	0.2	0.8370	0.7140	0.6183	0.5421	0.0135	-1.1806	5.1679	0.0880	0.0862
	0.5	0.7751	0.6182	0.5042	0.4189	0.0174	-0.8317	3.8878	0.1031	0.1018
	1	0.7225	0.5419	0.4188	0.3317	0.0200	-0.5708	3.2656	0.1126	0.1119
3	1.5	0.6898	0.4970	0.3708	0.2847	0.0212	-0.4215	3.0257	0.1167	0.1164
	2	0.6659	0.4654	0.3380	0.2536	0.0219	-0.3175	2.9062	0.1189	0.1187
	3	0.6317	0.4216	0.2942	0.2132	0.0226	-0.1738	2.8045	0.1209	0.1208
	4	0.6071	0.3914	0.2651	0.1871	0.0228	-0.0740	2.7767	0.1215	0.1215
	0.2	0.8225	0.6842	0.5746	0.4864	0.0078	-1.3076	5.8173	0.0661	0.0646
	0.5	0.7757	0.6119	0.4895	0.3963	0.0102	-0.9628	4.3773	0.0784	0.0772
	1	0.7354	0.5528	0.4233	0.3293	0.0120	-0.7087	3.6539	0.0865	0.0858
4	1.5	0.7101	0.5172	0.3849	0.2917	0.0130	-0.5653	3.3591	0.0904	0.0899
	2	0.6916	0.4918	0.3581	0.2661	0.0135	-0.4665	3.2013	0.0926	0.0924
	3	0.6647	0.4560	0.3214	0.2319	0.0142	-0.3317	3.0447	0.0951	0.0949
	4	0.6453	0.4309	0.2964	0.2092	0.0145	-0.2393	2.9764	0.0962	0.0962

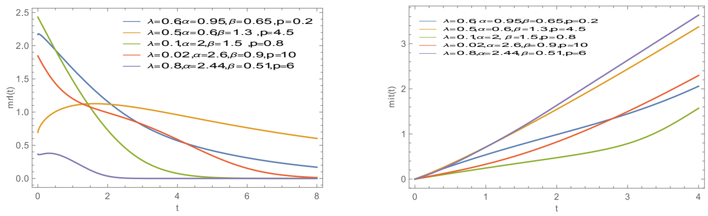


Fig. 4. The MRL and MIT for different parameters values

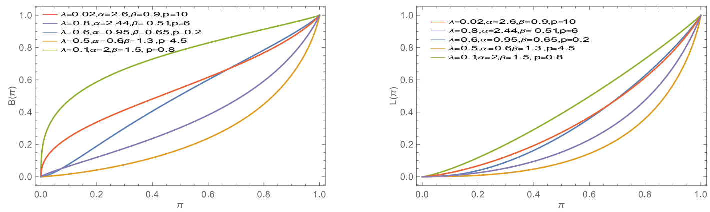


Fig. 5. Bonferroni and Lorenz curves for different parameters values

3.4 Recurrence relation for single moments of Generalized order statistics and characterization of GEP distribution

The concept of generalized order statistics (gos) was proposed by Kamps [23] as a unified concept describes ordered random variables and includes other models of ordered random variables as special models.

Consider a random sample of size n drawn from a population whose distribution function is $F(x)$, survival function $\bar{F}(x)$ and its density function $f(x)$. Then the statistic $X\,(s;n;m;k)$, $s=1,\ldots,n$, is said to be the s_{th} gos if its density function is given by

$$f_{X(s;n;m;k)}\,(x)=\frac{C_{s-1}}{(s-1)!}\;f\,(x)\;\;\left[\bar{F}\,(x)\right]^{\gamma_s-1}\;g_m^{s-1}\,(F\,(x))\,, \tag{3.4}$$

where $C_{s-1}=\prod\limits_{i=1}^{s}\gamma_i,\;\;i=1,\ldots,n-1,\;\;\gamma_i=k+(n-i)\,(m+1)\,,\;m,k$ are real numbers with $k>0$ and for $x\in(0,1)$.

$$g_m(x)=\begin{cases}\frac{1-(1-x)^{m+1}}{m+1}, & \text{if } m\neq-1\\ -log(1-x), & \text{if } m=-1.\end{cases}$$

The pdf in (3.4) is a special case of more general pdf considered by Kamps [23], in which the components of the vector $\tilde{m}=(m_1,\ldots,m_{n-1})$ are chosen such that $m_1=\ldots=m_{n-1}=m$. Hence $\gamma_i=k+n-i+\sum\limits_{j=i}^{n-1}m_j=k+(n-i)\,(m+1)$. The pdf in (3.4) is a special case of more general pdf considered by Kamps [23], in which the components of the vector $\tilde{m}=(m_1,\ldots,m_{n-1})$ are chosen such that $m_1=\ldots=m_{n-1}=m$. Hence $\gamma_i=k+n-i+\sum\limits_{j=i}^{n-1}m_j=k+(n-i)\,(m+1)$.

In order to establish the recurrence relation for single moments of gos from GEP distribution, the following lemma is considered, Athar and Islam [24].

Lemma 3.1. *Consider $\xi(x)$ is a measurable function of x and is differentiable,then*

$$E\left[\xi\left\{X\left(s;n;m;k\right)\right\}\right]-E\left[\xi\left\{X\left(s-1;n;m;k\right)\right\}\right]=\frac{C_{s-2}}{(s-1)!}\int_{0}^{\infty}\xi'\left(x\right)\;\;\left[\bar{F}\left(x\right)\right]^{\gamma_s}\;g_m^{s-1}\left(F\left(x\right)\right)dx$$

Theorem 3.2. *For the GEP distribution, $n\in\mathbb{N}$, $k>0$, $1\le s\le n$ and $r=1,2,\ldots$*

$$E\left[X^r\left(s;n;m;k\right)\right]-E\left[X^r\left(s-1;n;m;k\right)\right]=\frac{r}{p\alpha\lambda\beta\;\gamma_s}\;E\left[\phi\left\{X\left(s;n;m;k\right)\right\}\right] \tag{3.5}$$

where $\phi\left(x\right)=x^{r-\alpha}\;\frac{p+(1-p)e^{-\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}}{e^{\lambda x^{\alpha}}\left(e^{\lambda\;x^{\alpha}}-1\right)^{\beta-1}}$.

Proof. By lemma 3.1, let $\xi\left(x\right)=X^r$, then

$$E\left[X^r\left(s;n;m;k\right)\right]-E\left[X^r\left(s-1;n;m;k\right)\right]=\frac{C_{s-2}}{(s-1)!}\int_{0}^{\infty}r\;x^{r-1}\;\;\left[\bar{F}\left(x\right)\right]^{\gamma_s}\;g_m^{s-1}\left(F\left(x\right)\right)dx$$

The survival function of GEP distribution in (2.6) can be written as

$$\bar{F}\left(x\right)=\frac{p+(1-p)\,e^{-\left(e^{\lambda x^{\alpha}}-1\right)^{\beta}}}{p\alpha\lambda\beta x^{\alpha-1}e^{\lambda\;x^{\alpha}}\left(e^{\lambda x^{\alpha}}-1\right)^{\beta-1}}f\left(x\right)$$

then we have

$$E\left[X^r\left(s;n;m;k\right)\right]-E\left[X^r\left(s-1;n;m;k\right)\right]=$$
$$\frac{rC_{s-2}}{p\alpha\lambda\beta\;\;(s-1)!}\int_{0}^{\infty}\phi\left(x\right)\;\;f\left(x\right)\left[\bar{F}\left(x\right)\right]^{\gamma_s-1}\;g_m^{s-1}\left(F\left(x\right)\right)dx$$

Thus, we get

$$E\left[X^r\left(s,n,m,k\right)\right]-E\left[X^r\left(s-1,n,m,k\right)\right]=\frac{r}{p\alpha\lambda\beta\;\gamma_s}\;E\left[\phi\left\{X\left(s;n;m;k\right)\right\}\right]$$

□

Remark 3.1. Let $m=0$ and $k=1$ in (3.5), then the recurrence relation for single moments of order statistics of the GEP distribution is given as

$$E\left[X_{s:n}^r\right]-E\left[X_{s-1:n}^r\right]=\frac{r}{p\alpha\lambda\beta\left(n-s+1\right)}\;E\left[\phi\left\{X_{s:n}\right\}\right]$$

Remark 3.2. Let $m=-1$ and $k\ge1$ in (3.5), then the recurrence relation for single moments of k_{th} record values will be obtained.

A characterization result of GEP distribution is established based on recurrence relation (3.5). So that, the necessary and sufficient condition for a random variable X to be distributed with GEP distribution is (3.5), which can be proved as follows: if the recurrence relation in (3.5) is satisfied then using (3.4)

$$\frac{rC_{s-1}}{p\alpha\lambda\beta\gamma_s\ (s-1)!}\int_0^\infty \phi(x)\ f(x)\left[\bar{F}(x)\right]^{\gamma_s-1} g_m^{s-1}(F(x))\,dx$$

$$=\frac{C_{s-1}}{(s-1)!}\int_0^\infty x^r\ f(x)\left[\bar{F}(x)\right]^{\gamma_s-1} g_m^{s-1}(F(x))\,dx$$

$$-\frac{(s-1)\ C_{s-1}}{\gamma_s\ (s-1)!}\int_0^\infty x^r\ f(x)\left[\bar{F}(x)\right]^{\gamma_s+m} g_m^{s-2}(F(x))\,dx$$

$$=\frac{C_{s-1}}{(s-1)!}\int_0^\infty x^r\ f(x)\left[\bar{F}(x)\right]^{\gamma_s} g_m^{s-2}(F(x))\left\{\frac{g_m(F(x))}{\bar{F}(x)}-\frac{(s-1)\left[\bar{F}(x)\right]^m}{\gamma_s}\right\}dx$$

$$=\frac{C_{s-1}}{(s-1)!}\int_0^\infty x^r\ d\left\{-\frac{\left[\bar{F}(x)\right]^{\gamma_s} g_m^{s-1}(F(x))}{\gamma_s}\right\}$$

Integrating by parts, we obtain

$$\frac{rC_{s-2}}{p\alpha\lambda\beta\ (s-1)!}\int_0^\infty \phi(x)\ f(x)\left[\bar{F}(x)\right]^{\gamma_s-1} g_m^{s-1}(F(x))\,dx$$

$$=\frac{r\ C_{s-1}}{\gamma_s\ (s-1)!}\int_0^\infty x^{r-1}\ \left[\bar{F}(x)\right]^{\gamma_s} g_m^{s-1}(F(x))\,dx$$

which can be written as

$$\frac{r\ C_{s-1}}{\gamma_s\ (s-1)!}\int_0^\infty \left[\bar{F}(x)\right]^{\gamma_s-1} g_m^{s-1}(F(x))\left[x^{r-1}-\frac{\phi(x)\ f(x)}{p\alpha\lambda\beta\ \bar{F}(x)}\right]dx$$

An extension of M*ü*ntz-Sz*á*sz theorem, [25], can be applied to obtain

$$x^{r-1}-\frac{\phi(x)\ f(x)}{p\alpha\lambda\beta\ \bar{F}(x)}=0$$

thus

$$\bar{F}(x)=\frac{e^{-\left(e^{\lambda\ x^{\alpha}}-1\right)^{\beta}}}{p+(1-p)\,e^{-\left(e^{\lambda\ x^{\alpha}}-1\right)^{\beta}}}$$

4 Inference

This section provides estimation of the parameter vector $\Theta=(\lambda,\alpha,\beta,\text{ and }p)$ using maximum likelihood method. Asymptotic confidence intervals (CIs) for the model parameters are constructed based on the ML estimates.

4.1 Maximum likelihood estimation

In what follows, the maximum likelihood estimation for the model parameters are presented based on complete, type I censored, type II censored and random censored samples.

Case I: *Estimation from complete sample*

The maximum likelihood estimator for Θ is obtained from maximizing the likelihood function, or equivalently maximizing the log-likelihood function, corresponding to the GEP distribution. The log-likelihood function from complete sample is given by

$$\begin{aligned}\ell(\lambda,\ \alpha,\ \beta,p) =& n\ln(\lambda\ \alpha\ \beta\ p) - \sum_{i=1}^{n}\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta} + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) + \lambda\sum_{i=1}^{n}{x_i}^{\alpha}\\ &+(\beta-1)\sum_{i=1}^{n}\ln\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right) - 2\sum_{i=1}^{n}\ln\left(p+(1-p)\,\mathrm{e}^{-\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta}}\right)\end{aligned} \tag{4.1}$$

Case II: *Estimation from type I censored sample*

Consider a predetermined time x_c represents the duration of studying the lifetime of n units and suppose that $d < n$ is the observed lifetimes until the time x_c, then there are $(n-d)$ censored times. The log-likelihood function in this case is given by

$$\begin{aligned}\ell(\lambda,\ \alpha,\ \beta,p) =& r\ln(\lambda\ \alpha\ \beta\ p) - \sum_{i\in T}\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta} + (\alpha-1)\sum_{i\in T}\ln(x_i) + \lambda\sum_{i\in T}{x_i}^{\alpha} + \ln(n!) - \ln((n-r)!)\\ &+(\beta-1)\sum_{i\in T}\ln\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right) - 2\sum_{i\in T}\ln\left(p+(1-p)\,\mathrm{e}^{-\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta}}\right) - (\mathrm{n}-\mathrm{d})\left(\mathrm{e}^{\lambda {x_c}^{\alpha}}-1\right)^{\beta}\\ &-(\mathrm{n}-\mathrm{d})\ln\left(p+(1-p)\,\mathrm{e}^{-\left(\mathrm{e}^{\lambda {x_c}^{\alpha}}-1\right)^{\beta}}\right)\end{aligned} \tag{4.2}$$

where, T denotes the set of non-censored observations with d lifetimes.

Case III: *Estimation from type II censored sample*

In this case, the study continues until the failure of the first r units, $r < n$. The log-likelihood function, in this case, is given by

$$\begin{aligned}\ell(\lambda,\ \alpha,\ \beta,p) =& r\ln(\lambda\ \alpha\ \beta\ p) - \sum_{i\in T}\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta} + (\alpha-1)\sum_{i\in T}\ln(x_i) + \lambda\sum_{i\in T}{x_i}^{\alpha} + \ln(n!) - \ln((n-r)!)\\ &+(\beta-1)\sum_{i\in T}\ln\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right) - 2\sum_{i\in T}\ln\left(p+(1-p)\,\mathrm{e}^{-\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta}}\right)\\ &-(\mathrm{n}-\mathrm{r})\left(\mathrm{e}^{\lambda {x_{(r)}}^{\alpha}}-1\right)^{\beta} - (\mathrm{n}-\mathrm{r})\ln\left(p+(1-p)\,\mathrm{e}^{-\left(\mathrm{e}^{\lambda {x_{(r)}}^{\alpha}}-1\right)^{\beta}}\right),\end{aligned} \tag{4.3}$$

where $x_{(r)}$ is the ordinary order statistic and T denotes the set of non-censored observations with r lifetimes.

Case IV: *Estimation from random censored sample*

Random censoring is the more general scheme. Consider the observed times are $X_i = \min(T_i, C_i)$, $i = 1, \ldots, n$, where T_i is the lifetime for the i_{th} individual, assumed to have the GEP distribution, and C_i is the censoring time for the i_{th} individual, assumed to have a non-informative distribution, i.e. a distribution that does not involve the parameters λ,α,β and p. The T_i and C_is are assumed independent. In this case, the log-likelihood function is given by

$$\begin{aligned}\ell(\lambda,\ \alpha,\ \beta,p) =& rln(\lambda\ \alpha\ \beta\ p) - \sum_{i=1}^{n}\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta} + (\alpha-1)\sum_{i\in T}\ln(x_i) + \lambda\sum_{i\in T}{x_i}^{\alpha}\\ &+(\beta-1)\sum_{i\in T}\ln\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right) - 2\sum_{i\in T}\ln\left(p+(1-p)\,\mathrm{e}^{-\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta}}\right)\\ &-\sum_{i\in C}\ln\left(p+(1-p)\,\mathrm{e}^{-\left(\mathrm{e}^{\lambda {x_i}^{\alpha}}-1\right)^{\beta}}\right)\end{aligned} \tag{4.4}$$

where, T denotes the set of non-censored observations with r lifetimes and C denotes the set of censored observations with $n-r$ censored times.

For any of the above cases, the score functions are the partial derivatives ($\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}$ and $\frac{\partial \ell}{\partial p}$) of the log-likelihood functions with respect to λ,α,β and p. The maximum likelihood estimate (MLE)$\hat{\Theta} = (\hat{\lambda}, \hat{\alpha}, \hat{\beta}, \hat{p})$ of $\Theta = (\lambda, \alpha, \beta, p)$ is obtained by solving the nonlinear equations $\frac{\partial \ell}{\partial \lambda} = 0, \frac{\partial \ell}{\partial \alpha} = 0, \frac{\partial \ell}{\partial \beta} = 0$ and $\frac{\partial \ell}{\partial p} = 0$ simultaneously. These equations can not be solved analytically so explicit expressions for the MLEs of λ, α, β and p are not available. Hence, these equations can be solved numerically via iterative methods such as Newton-Raphson technique using mathematical packages.

4.2 Confidence intervals

The normal approximation of the estimated Θ can be used to construct approximate confidence intervals and for testing the hypotheses on the parameters λ, α, β and p. Under conditions that are fulfilled for the parameters in the interior of the parameter space, the asymptotic distribution of $\sqrt{n}\left(\hat{\Theta} - \Theta\right)$ is $\mathcal{N}_4(0, I^{-1}(\Theta))$, where $I(\Theta)$ is the expected Fisher information matrix. This asymptotic behavior remains valid if $I(\Theta)$ is replaced by the observed information matrix evaluated at $\hat{\Theta}$, that is $J(\hat{\Theta})$. The observed information matrix $J(\Theta)$ is given by

$$J(\Theta) = -\begin{pmatrix} L_{\lambda\lambda} & L_{\lambda\alpha} & L_{\lambda\beta} & L_{\lambda p} \\ L_{\alpha\lambda} & L_{\alpha\alpha} & L_{\alpha\beta} & L_{\alpha p} \\ L_{\beta\lambda} & L_{\beta\alpha} & L_{\beta\beta} & L_{\beta p} \\ L_{p\lambda} & L_{p\alpha} & L_{p\beta} & L_{pp} \end{pmatrix}$$

where the components of $J(\Theta)$ are the second derivative of the log-likelihood function with respect to λ, α, β and p. The approximate $100(1-\nu)\%$ confidence intervals for the parameters λ, α, β and p. are $\hat{\lambda} \pm \mathcal{Z}_{\frac{\nu}{2}} \sqrt{Var\left(\hat{\lambda}\right)}$, $\hat{\alpha} \pm \mathcal{Z}_{\frac{\nu}{2}} \sqrt{Var\left(\hat{\alpha}\right)}$, $\hat{\beta} \pm \mathcal{Z}_{\frac{\nu}{2}} \sqrt{Var\left(\hat{\beta}\right)}$ and $\hat{p} \pm \mathcal{Z}_{\frac{\nu}{2}} \sqrt{Var\left(\hat{p}\right)}$ respectively, where $Var(\hat{\lambda})$, $Var(\hat{\alpha})$,$Var(\hat{\beta})$ and $Var(\hat{p})$ are the diagonal elements of $I^{-1}(\hat{\Theta})$ corresponding to each parameter and $\mathcal{Z}_{\frac{\nu}{2}}$ is the upper $\frac{\nu}{2}$ percentile of standard normal distribution.

5 Simulation Study

In this section, a Monte Carlo simulation study is performed in order to examine the accuracy of the MLEs of the parameters of the GEP distribution. Observations are generated from the GEP distribution using the inverse transformation method for different parameter combinations. The simulation study is repeated 1000 times each with sample size $n = 25, 50, 100, 150, 200, 400$ and 800 and parameter vectors $\Theta = (\lambda, \alpha, \beta, p)$= (0.6,0.95,0.65,0.12), (0.5,2.5,1.5,2) and (1,0.4,0.8,0.2). These selected values of Θ gives decreasing-increasing-decreasing, unimodal and decreasing shapes for the density function of GEP distribution. Table 2 shows the results of the simulations corresponding to complete samples. These results verify that mean square error(MSE) and average bias (ABS) decay toward zero when the sample size n increases. The type I, type II and random censored samples are generated at censoring percentage 10 % and 30% of the sample size. For type I censored samples the simulation was carried with ending times $x_c = 3$ and $x_c = 6$. Tables 3, 4 and 5 show the simulation results based on type I, type II and random censored samples, respectively. We noticed from these results that MSEs and average bias tends to decrease as the sample size increases.

Table 2. MSEs and ABs for the model parameters in case of complete samples

n	$\lambda = 0.6$		$\alpha = 0.95$		$\beta = 0.65$		$p = 0.12$	
	MSEs	ABs	MSE	ABs	MSEs	ABs	MSE	ABs
25	0.00213	0.00746	0.00295	0.01023	0.00335	0.00991	0.30731	-0.00433
50	0.00097	0.00370	0.00132	0.00563	0.00150	0.00534	0.00874	0.00058
100	0.00050	0.00174	0.00067	0.00262	0.00076	0.00252	0.00045	0.00149
150	0.00032	0.00125	0.00043	0.00175	0.00049	0.00174	0.00031	0.00127
200	0.00012	0.00085	0.00033	0.00124	0.00037	0.00114	0.00022	0.00086
400	0.00024	0.00057	0.00016	0.00078	0.00018	0.00078	0.00011	0.00051
800	0.00006	0.00017	0.00008	0.00021	0.00009	0.00020	0.00005	0.00026
n	$\lambda = 0.5$		$\alpha = 2.5$		$\beta = 1.5$		$p = 2$	
	MSEs	ABs	MSE	ABs	MSEs	ABs	MSE	ABs
25	0.0035	0.0154	0.1607	0.1136	0.0643	0.0493	0.5892	0.1600
50	0.0015	0.0074	0.0595	0.0459	0.0243	0.0159	0.2599	0.0791
100	0.0008	0.0026	0.0271	0.0223	0.0122	0.0100	0.1314	0.0286
150	0.0005	0.0018	0.0189	0.0188	0.0083	0.0092	0.0819	0.0128
200	0.0004	0.0023	0.0133	0.0157	0.0063	0.0074	0.0641	0.0275
400	0.0002	0.0012	0.0069	0.0059	0.0033	0.0018	0.0310	0.0147
800	0.0001	0.0006	0.0035	0.0015	0.0016	-0.0005	0.0142	0.0105
n	$\lambda = 1$		$\alpha = 0.4$		$\beta = 0.8$		$p = 0.2$	
	MSEs	ABs	MSE	ABs	MSEs	ABs	MSE	ABs
25	0.0062	0.0118	0.0026	0.0097	0.0068	0.0125	0.0058	0.0156
50	0.0033	0.0065	0.0011	0.0048	0.0034	0.0063	0.0029	0.0085
100	0.0017	0.0031	0.0006	0.0024	0.0018	0.0035	0.0013	0.0044
150	0.0010	0.0028	0.0003	0.0025	0.0010	0.0034	0.0008	0.0017
200	0.0008	0.0005	0.0003	0.0012	0.0008	0.0011	0.0006	0.0004
400	0.0004	0.0012	0.0001	0.0009	0.0004	0.0012	0.0003	0.0010
800	0.0002	0.0011	0.0001	0.0008	0.0002	0.0014	0.0002	0.0004

Table 3. MSEs and ABs estimates for the model parameters in case of 10% and 30% type I censored samples with $x_c = 3$ and $x_c = 6$

x_c	censoring	n	$\lambda = 1$		$\alpha = 0.4$		$\beta = 0.8$		$p = 0.2$	
			MSE	ABs	MSE	ABs	MSE	ABs	MSE	ABs
3	10%	10	0.0089	0.0640	0.0094	0.0778	0.0140	0.0904	0.0320	0.0905
		30	0.0042	0.0530	0.0071	0.0786	0.0088	0.0848	0.0083	0.0585
		50	0.0036	0.0530	0.0071	0.0810	0.0085	0.0867	0.0060	0.0547
		100	0.0029	0.0510	0.0068	0.0804	0.0077	0.0845	0.0037	0.0492
		200	0.0028	0.0510	0.0068	0.0814	0.0076	0.0857	0.0029	0.0484
3	30%	10	0.0127	-0.1122	0.0030	-0.0249	0.0095	-0.0806	0.0484	0.1258
		30	0.0127	-0.1120	0.0013	-0.0254	0.0076	-0.0823	0.0156	0.0971
		50	0.0126	-0.1117	0.0011	-0.0265	0.0074	-0.0831	0.0137	0.0969
		100	0.0126	-0.1116	0.0008	-0.0237	0.0068	-0.0808	0.0093	0.0867
		200	0.0126	-0.1081	0.0007	-0.0241	0.0067	-0.0810	0.0084	0.0863
6	10%	10	0.0110	-0.0835	0.0041	-0.0521	0.0119	-0.0880	2.7674	0.1313
		30	0.0110	-0.0973	0.0031	-0.0521	0.0102	-0.0940	0.0052	0.0159
		50	0.0107	-0.0992	0.0030	-0.0523	0.0099	-0.0951	0.0028	0.0110
		100	0.0106	-0.1008	0.0029	-0.0523	0.0096	-0.0962	0.0014	0.0064
		200	0.0103	-0.1006	0.0028	-0.0526	0.0095	-0.0965	0.0007	0.0069
6	30%	10	0.0829	-0.2922	0.0229	-0.1493	0.0747	-0.2708	0.0356	0.0778
		30	0.0848	-0.2923	0.0218	-0.1469	0.0732	-0.2699	0.0073	0.0254
		50	0.0856	-0.2924	0.022	-0.1481	0.0741	-0.2718	0.0046	0.0239
		150	0.0855	-0.293	0.0219	-0.1478	0.0739	-0.2716	0.0022	0.0186
		200	0.0854	-0.2972	0.0218	-0.1475	0.0736	-0.2712	0.001	0.0149

Table 4. MSEs and ABs estimates for the model parameters in case of 10% and 30% type II censored samples

censoring	n	$\lambda = 0.5$		$\alpha = 2.5$		$\beta = 1.5$		$p = 2$	
		MSE	ABs	MSE	ABs	MSE	ABs	MSE	ABs
10%	10	0.0153	-0.0211	0.5132	-0.0491	0.1515	-0.0218	3.13	0.726
	30	0.0113	-0.0841	0.4176	-0.446	0.1283	-0.2135	0.562	0.166
	50	0.0105	-0.0958	0.3195	-0.5272	0.0937	-0.2659	0.307	0.096
	100	0.0092	-0.1131	0.2839	-0.6293	0.0928	-0.3392	0.136	0.029
	200	0.0087	-0.1221	0.265	-0.6823	0.0881	-0.3798	0.0755	0.003
30%	10	0.0578	-0.1357	1.5881	-0.5191	0.5593	-0.1549	5.83	1.057
	30	0.0523	-0.1979	1.4213	-0.9918	0.4724	-0.4971	0.795	0.236
	50	0.0455	-0.211	1.2196	-1.0893	0.3716	-0.5853	0.462	0.173
	100	0.0408	-0.2271	1.045	-1.1838	0.2974	-0.6748	0.205	0.065
	200	0.0272	-0.2397	0.844	-1.2567	0.2959	-0.7427	0.092	0.05
censoring	n	$\lambda = 1$		$\alpha = 0.4$		$\beta = 0.8$		$p = 0.2$	
		MSE	ABs	MSE	ABs	MSE	ABs	MSE	ABs
10%	10	0.039	-0.059	0.0116	-0.0237	0.0377	-0.0527	0.0296	0.0672
	30	0.0314	-0.1367	0.01	-0.0745	0.0305	-0.1342	0.0049	0.0167
	50	0.0266	-0.1511	0.0095	-0.0822	0.0268	-0.1478	0.0031	0.0078
	100	0.0262	-0.1717	0.0078	-0.095	0.0254	-0.1697	0.0015	0.006
	200	0.0244	-0.1943	0.007	-0.1064	0.023	-0.1914	0.0006	0.0026
30%	10	0.1673	-0.209	0.0381	-0.0869	0.1365	-0.1812	0.0373	0.0904
	30	0.1473	-0.3157	0.0344	-0.1566	0.1215	-0.2907	0.0092	0.0314
	50	0.124	-0.3465	0.0296	-0.17	0.1035	-0.3171	0.0044	0.0144
	100	0.1061	-0.3803	0.0258	-0.1844	0.0893	-0.3461	0.0019	0.0066
	200	0.0624	-0.4066	0.0214	-0.1946	0.0564	-0.3678	0.001	0.0059

Table 5. MSEs and ABs estimates for the model parameters in case of 10% and 30% random censored samples

censoring	n	$\lambda = 0.5$		$\alpha = 2.5$		$\beta = 1.5$		$p = 2$	
		MSE	ABs	MSE	ABs	MSE	ABs	MSE	ABs
10%	10	0.0082	-0.0614	0.6584	0.2362	0.2357	0.1609	25.4870	2.3788
	30	0.0055	-0.0634	0.1547	0.0348	0.0869	0.1005	10.5811	1.8292
	50	0.0054	-0.0663	0.0698	-0.0386	0.0496	0.0740	4.6435	1.5588
	100	0.0045	-0.0637	0.0316	-0.0198	0.0305	0.1067	2.4563	1.3225
	200	0.0045	-0.0653	0.0149	-0.0412	0.0195	0.1037	1.7589	1.2300
30%	10	0.0419	-0.1990	0.4525	0.1174	0.7744	0.6150	24.2980	7.6671
	30	0.0354	-0.1859	0.1053	-0.1244	0.3432	0.4398	18.0970	5.0298
	50	0.0332	-0.1804	0.0831	-0.1682	0.2627	0.3854	7.4370	2.6214
	100	0.0322	-0.1785	0.0589	-0.1993	0.1698	0.3481	2.0630	0.9823
	200	0.0319	-0.1783	0.0486	-0.1987	0.1466	0.3493	1.4791	0.8514
censoring	n	$\lambda = 1$		$\alpha = 0.4$		$\beta = 0.8$		$p = 0.2$	
		MSE	ABs	MSE	ABs	MSE	ABs	MSE	ABs
10%	10	0.0556	0.0395	0.0353	0.0968	0.1111	0.1420	0.0823	0.1959
	30	0.0315	-0.0215	0.0145	0.0497	0.0477	0.0585	0.0299	0.1173
	50	0.0217	-0.0678	0.0068	0.0183	0.0217	0.0008	0.0180	0.0868
	100	0.0199	-0.1021	0.0033	-0.0028	0.0119	-0.0393	0.0090	0.0588
	200	0.0193	-0.1215	0.0021	-0.0062	0.0098	-0.0636	0.0057	0.0461
30%	10	0.1634	-0.2518	0.0501	0.1134	0.1155	0.0940	1.4221	0.6615
	30	0.1558	-0.3486	0.0118	0.0119	0.0385	-0.0708	0.2340	0.3944
	50	0.1395	-0.3627	0.0066	-0.0051	0.0287	-0.1029	0.1490	0.3314
	100	0.1361	-0.3900	0.0040	-0.0315	0.0303	-0.1488	0.0959	0.2740
	200	0.1044	-0.4016	0.0037	-0.0438	0.0333	-0.1693	0.0682	0.2377

6 Application

In this section, three real data sets, two with censored data, are used to show the potentiality of the GEP distribution in modeling positive data. The GEP distribution is fitted to the data sets. For the comparison, the following models are considered as competitive for GEP distribution:

- Complementary exponential power (CEP) distribution [2] with density function given by
$f_{CEP}(x) = \frac{\beta\theta x^{\beta-1}}{\alpha^{\beta}} \ \exp\left(1 + \left(\frac{x}{\alpha}\right)^{\beta} - e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right)\left[1 - \exp\left(1 - e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right)\right]^{\theta-1}$
- Complementary exponentiated exponential geometric (CEEG) distribution [26] with density function given by
$f_{CEEG}(x) = \alpha\lambda p \mathrm{e}^{-x\lambda}\left(1 - \mathrm{e}^{-x\lambda}\right)^{\alpha-1}\left[1 - (1-p)\left(1 - \mathrm{e}^{-x\lambda}\right)^{\alpha}\right]^{-2}$
- The exponentiated power generalized Weibull (EPGW) distribution [27] with density function given by
$f_{EPGW}(x) = \alpha\beta\gamma\lambda\ x^{\gamma-1}\ (1 + \lambda\ x^{\gamma})^{\alpha-1}\mathrm{e}^{1-(1+x^{\gamma}\lambda)^{\alpha}}\left(1 - \mathrm{e}^{1-(1+x^{\gamma}\lambda)^{\alpha}}\right)^{\beta-1}$
- A modified distribution referred to as MDAL distribution introduced by Tafakori [28] with density function given by
$f_{MDAL}(x) = \left(\frac{-\alpha\beta\lambda}{\ln p}x^{\beta-1}\left(1 + \lambda\ x^{\beta}\right)^{\alpha-1}\right)\left(\frac{(1-p)\exp\left[1-\left(1+x^{\beta}\lambda\right)^{\alpha}\right]}{1-(1-p)\exp\left[1-\left(1+x^{\beta}\lambda\right)^{\alpha}\right]}\right)$

For the data sets, the MLEs of the parameters of the selected distributions is obtained. To choose the best possible model, we obtained Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov-Smirnov (KS) distances between the empirical distribution function and the fitted distribution function (as well as its respective P-value), AndersonDarling statistic (A^*) and Cramrvon Mises statistic (W^*). The statistics A^* and W^* are described in details by Chen and Balakrishnan [29]. The probability-probability (p-p) plots for the fitted models for the data sets are presented. For the p-p plots for complete data, we plotted $F(x_{(i)};\hat{\Theta})$ against $(\frac{i-c}{n+1-2c}), i = 1, 2, \ldots, n$, where $x_{(i)}$ are the ordered values of the observed data. For censored data, an analogous modification of Kaplan-Meier estimator considered the plotting position (see Waller and Turnbull [30])

$$p\left(x_i\right) = 1 - \frac{n-c+1}{n-2c+1}\prod_{j\in S, j\leq i}\frac{n-j-c+1}{n-j-c+2} \tag{6.1}$$

where S is the set of all subscripts j such that x_j is an observed failure time and $0 \leq c \leq 1$, we set $c = \frac{3}{8}$ (see Harter [31]). As a measure of closeness to the diagonal line, the sum of squared (SS) difference between observed and expected probability is computed. The model with less values of AIC, BIC, SS, K-S, W^* and A^* is considered the best fit for a data set. The required numerical evaluations are implemented using Mathematica software.

6.1 Aarset data

The Aarset data set describes lifetimes of 50 industrial devices on life testing at a time zero [32]. The fitting results for GEP distribution and the competitive models, selected in this paper, are presented in table 6. To show that the likelihood functions have a unique solution, the profiles of the log-likelihood function of α, β, λ and p are plotted in fig. 6.

Furthermore, visual comparison using Kaplan-Meier curve and the fitted survival curves and empirical and fitted densities in figure 7 and Q-Q plots for the fitted distributions in fig. 8. Table 6. and figs (7, 8) show a very strong evidence for the superiority GEP distribution to the other models.

Table 6. MLEs of the parametrs discrimination criteria for Aarset data

Model	MLEs	AIC	BIC	SS	K-S (p-value)	W^*	A^*
GEP	$\hat{\lambda}$=5.046 × 10^{-9}, $\hat{\alpha}$= 4.85, $\hat{\beta}$=0.0998, $\hat{p}$= 0.51.	437.781	445.429	0.0902	0.127 (0.3958)	0.0907	0.8059
EP	$\hat{\lambda}$= 0.659, $\hat{\alpha}$= 0.0117.	473.861	477.685	0.4932			
CEP	$\hat{\alpha}$=27.97, $\hat{\beta}$=0.162, $\hat{\theta}$= 16.72.	461.522	467.258	0.5823	0.2109 (0.0234)	0.5848	3.656
CEEG	$\hat{\lambda}$=0.036, $\hat{\alpha}$= 0.2569, $\hat{p}$= 0.0675.	474.175	479.911	0.302	0.151 (0.2041)	0.3045	2.1719
EPGW	$\hat{\lambda}$=5.9 × 10^{-5}, $\hat{\alpha}$= 1.2923, $\hat{\beta}$=0.4262, $\hat{\gamma}$= 2.1777.	477.284	484.932	0.4492	0.1776 (0.0853)	0.45005	3.6955
MDAL	$\hat{\lambda}$=0.0107, $\hat{\alpha}$= 14.8816, $\hat{\beta}$=0.5662, $\hat{p}$= 172.357.	464.641	472.289	0.3091	0.1376 (0.3003)	0.309	2.4157

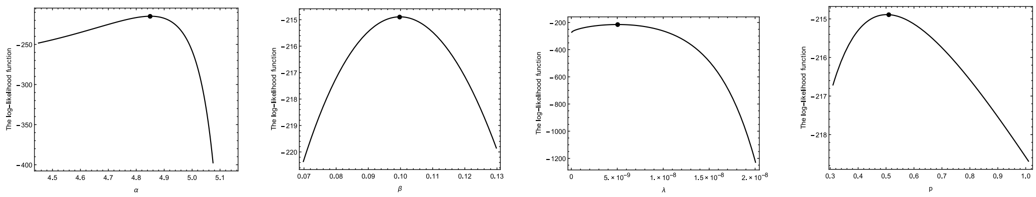
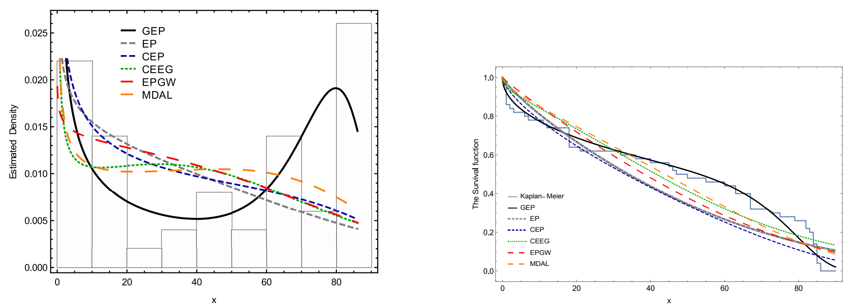


Fig. 6. For Aarset data: The profiles of the log-likelihood function of α, β, λ and p



(a) (b)

Fig. 7. For Aarset data: (a) Empirical and fitted densities (b) Empirical and fitted survival functions

6.2 Bladder cancer data

The second data set, with random censoring mechanism, consists of the remission times (in months) of 137 bladder cancer patients with nine censored times [33]. The fitting results for GEP distribution and the competitive models, selected in this paper, are presented in table 7. To show that the likelihood functions have a unique solution, the profiles of the log-likelihood function of α, β, λ and p are plotted in fig. 9.

Furthermore, visual comparison using Kaplan-Meier curve and the fitted survival curves in fig. 10 and p-p plots for the fitted distributions in fig. 11. These results show a strong evidence for the superiority GEP distribution to the other models.

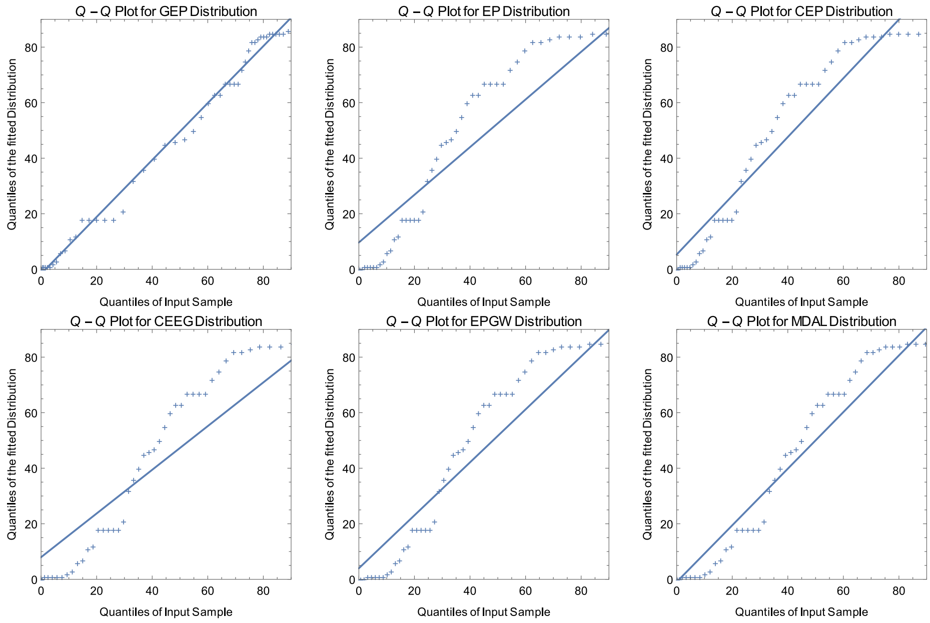


Fig. 8. Q-Q plots for the fitted distributions for Aarset data

Table 7. MLEs of the parametrs discrimination criteria for bladder cancer data

Model	MLEs	AIC	BIC	SS	K-S (p-value)	W^*	A^*
GEP	$\hat{\lambda}$=0.2076, $\hat{\alpha}$= 0.3016, $\hat{\beta}$=4.5935, $\hat{p}$= 42.474.	844.575	856.255	0.01225	0.0462 (0.9321)	0.0504	0.3273
EP	$\hat{\lambda}$= 0.1337, $\hat{\alpha}$= 0.6733.	872.544	878.384	0.5393	0.133 (0.0157)	0.7922	4.4975
CEP	$\hat{\alpha}$=2.3641, $\hat{\beta}$=0.2394, $\hat{\theta}$= 7.9152.	846.695	855.455	0.1644	0.0665 (0.5801)	0.1069	0.6237
CEEG	$\hat{\lambda}$=0.0658, $\hat{\alpha}$= 1.6238, $\hat{p}$= 4.4606.	843.583	852.343	0.0221	0.0557 (0.7884)	0.0805	0.4948
EPGW	$\hat{\lambda}$=0.3289, $\hat{\alpha}$= 0.6519, $\hat{\beta}$=1.7069, $\hat{\gamma}$= 0.9753.	846.297	857.977	0.0362	0.0662 (0.5847)	0.1135	0.6016
MDAL	$\hat{\lambda}$=0.0914, $\hat{\alpha}$= 0.444, $\hat{\beta}$=1.5523, $\hat{p}$= 0.5718.	845.573	857.253	0.0256	0.0608 (0.6927)	0.0931	0.5163

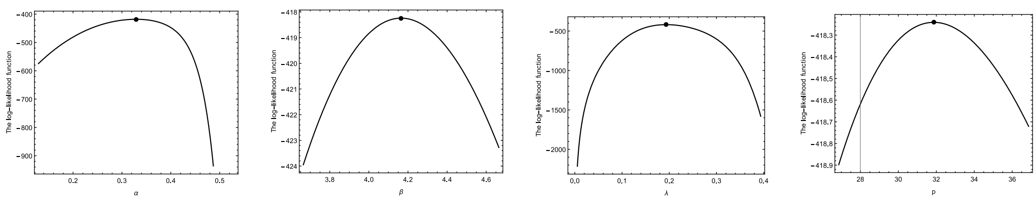


Fig. 9. For bladder cancer data: The profiles of the log-likelihood function of α, β, λ and p

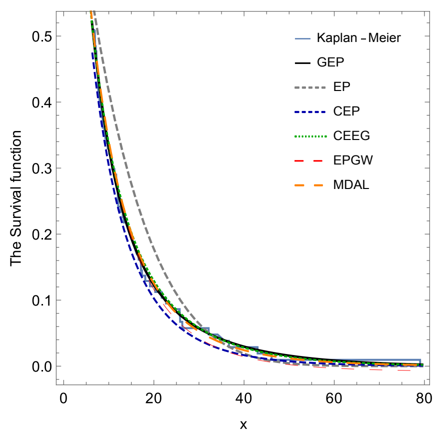


Fig. 10. Empirical and fitted survival functions for bladder cancer data

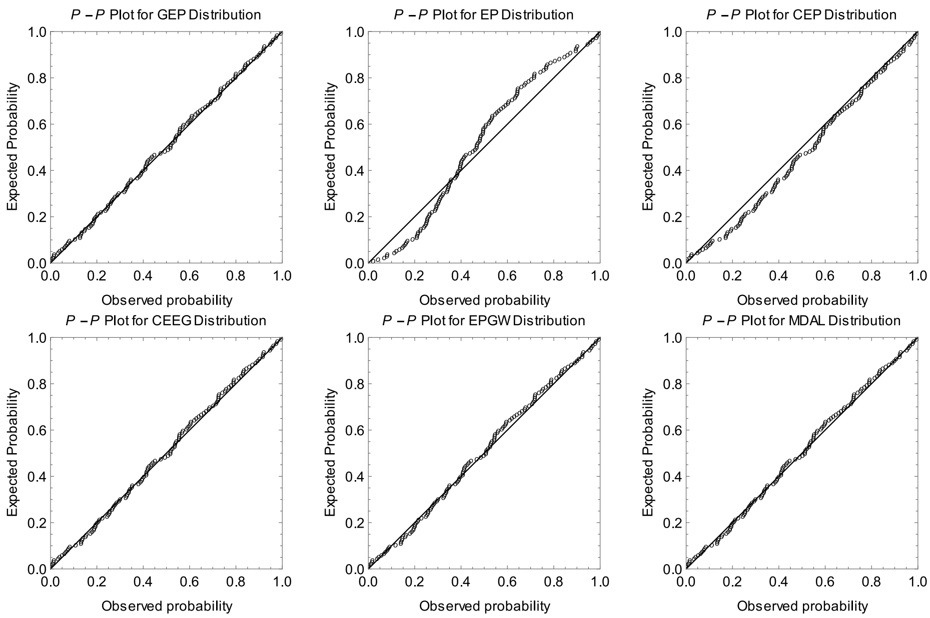


Fig. 11. p-p plots for the fitted distributions for bladder cancer data

6.3 Remission times

The last data set available in Bain and Engelhardt [34]. This data set represents the remission times of forty leukemia patients due to administrating a new drug. The experiment was terminated after 7 months (210 days). The number of observations during the experiment were 22 remission times. Clearly, it is a Type-I censored sample with $n = 40$ and $d = 22$. For type I censored data we consider AIC and modifications of AndersonDarling statistic (A^{**}) and Cramrvon Mises statistic (W^{**}) as discrimination criteria. These criteria were discussed in details by D'Agostino and Stephens [35]. Comparing the empirical survival function with the fitted survivals in fig. (12). From the results in table 7. it can be observed that GEP distribution provides a better fit to thee data set.

Table 8. MLEs of parametrs, AIC, A^{**} and W^{**} for remission times data

Model	MLEs	AIC	A^{**}	W^{**}
GEP	$\hat{\lambda}$=0.0055, $\hat{\alpha}$= 0.7959, $\hat{\beta}$=2.7942, $\hat{p}$= 8.821.	291.466	0.2056	0.0269
EP	$\hat{\lambda}$= 0.008, $\hat{\alpha}$= 0.7921.	301.534	1.75421	0.2902
CEP	$\hat{\alpha}$=78.6118, $\hat{\beta}$=0.5825, $\hat{\theta}$= 8.8172.	343.251	20.755	3.4995
CEEG	$\hat{\lambda}$=0.0256, $\hat{\alpha}$= 3.565, $\hat{p}$= 0.0235.	293.702	0.8941	0.0633
EPGW	$\hat{\lambda}$=3.6679, $\hat{\alpha}$=0.143, $\hat{\beta}$=16.0951, $\hat{\gamma}$= 1.5913.	298.59	0.4892	0.0746
MDAL	$\hat{\lambda}$=0.003, $\hat{\alpha}$= 0.7428, $\hat{\beta}$=0.8907, $\hat{p}$= 0.1365.	314.298	1.4181	0.2101

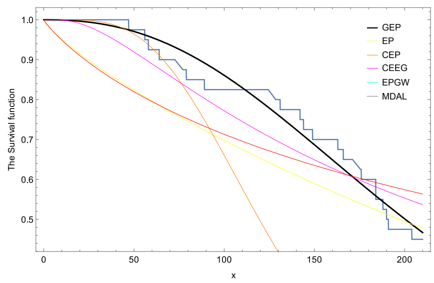


Fig. 12. Empirical and fitted survival functions for remission times data

7 Conclusion

In this paper, a generalization of the exponential power distribution is generated, on the latent of complementary risks problem and frailty models, with two extra shape parameters. The hazard function of the new distribution is more flexible than EP distribution. Some statistical and reliability properties of the GEP distribution are presented. Estimations of the model parameters have provided using maximum likelihood method based on complete, type I, type II and random censored samples. Monte Carlo simulations are carried out to compare the long-run performance of the maximum likelihood estimators of the model parameters. At the end of this paper the proposed model has been compared to other alternative models by fitting these models to real data sets; and the results showed that the GEP distribution provides a better fit whether the data are examples of complete sample, random or type I censored samples.

Competing Interests

The authors declare that no competing interests exist.

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