

ON THE MULTIPLICATIVE DEGREE BASED TOPOLOGICAL INDICES OF CIRCULANT GRAPH

GHULAM HUSSAIN¹, SABA NOREEN, WASEEM KHAILD

ABSTRACT. Topological indices are numerical numbers associated with a graph that helps to predict many properties of underlined graph. In this paper we aim to compute multiplicative topological indices of Circulant graph.

Key words and phrases: Zagreb index; Randić index; Polynomial; Degree; Graph.

1. Introduction

A number, polynomial or a matrix can uniquely identify a graph. A topological index is a numeric number associated to a graph which completely describes the topology of the graph, and this quantity is invariant under the isomorphism of graphs. The degree-based topological indices are derived from degrees of vertices in the graph. These indices have many correlations to chemical properties. In other words, a topological index remains invariant under graph isomorphism. The study of topological indices, based on distance in a graph, was effectively employed in 1947 in chemistry by Weiner [1]. He introduced a distance-based topological index called the "Wiener index" to correlate properties of alkenes and the structures of their molecular graphs. Recent progress in nano-technology is attracting attention to the topological indices of molecular graphs, such as nanotubes, nanocones, and fullerenes to cut short experimental labor. Since their introduction, more than 140 topological indices have been developed, and experiments reveal that these indices, in combination, determine the material properties such as melting point, boiling point, heat of formation, toxicity, toughness,

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¹ Corresponding Author

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and stability [2]. These indices play a vital role in computational and theoretical aspects of chemistry in predicting material properties [3-8].

Several algebraic polynomials have useful applications in chemistry, such as the Hosoya Polynomial (also called the Wiener polynomial) [9]. It plays a vital role in determining distance-based topological indices. Among other algebraic polynomials, the M polynomial introduced recently in 2015 [10] plays the same role in determining the closed form of many degree-based topological indices. Other famous polynomials are the first Zagreb polynomial and the second Zagreb polynomial.

A graph G is an ordered pair (V, E) , where V is the set of vertices and E is the set of edges. A path from a vertex v to a vertex w is a sequence of vertices and edges that starts from v and stops at w . The number of edges in a path is called the length of that path. A graph is said to be connected if there is a path between any two of its vertices. The distance $d(u, v)$ between two vertices u and v of a connected graph G is the length of a shortest path between them. Graph theory is contributing a lion's share in many areas such as chemistry, physics, pharmacy, as well as in industry [11]. We will start with some preliminary facts. The first and second multiplicative Zagreb indices [12], respectively

$$MZ_1(G) = \prod_{u \in V(G)} (d_u)^2,$$

$$MZ_2(G) = \prod_{uv \in E(G)} d_u \cdot d_v,$$

and the Narumi-Katayama index [13]

$$NK(G) = \prod_{u \in V(G)} d_u.$$

Like the Wiener index, these types of indices are the focus of considerable research in computational chemistry [14-17]. For example, in 2011, Gutman [14] characterized the multiplicative Zagreb indices for trees and determined the unique trees that obtained maximum and minimum values for $M_1(G)$ and $M_2(G)$, respectively. Wang *et al.* [17] then extended Gutman's result to the following index for k-trees,

$$W_1^s(G) = \prod_{u \in V(G)} (d_u)^s.$$

Notice that $s = 1, 2$ is the Narumi-Katayama and Zagreb index, respectively. Based on the successful consideration of multiplicative Zagreb indices, Eliasi *et al.* [18] continued to define a new multiplicative version of the first Zagreb index as

$$MZ_1^*(G) = \prod_{uv \in E(G)} (d_u + d_v).$$

Furthering the concept of indexing with the edge set, the researchers introduced the first and second hyper-Zagreb indices of a graph [19] as

$$HII_1(G) = \prod_{uv \in E(G)} (d_u + d_u)^2,$$

$$HII_2(G) = \prod_{uv \in E(G)} (d_u \cdot d_u)^2.$$

In [20], Kulli *et al.* defined the first and second generalized Zagreb indices

$$MZ_1^a(G) = \prod_{uv \in E(G)} (d_u + d_u)^a,$$

$$MZ_2^a(G) = \prod_{uv \in E(G)} (d_u \cdot d_u)^a.$$

Multiplicative sum connectivity and multiplicative product connectivity indices [21] are define as:

$$SCII(G) = \prod_{uv \in E(G)} \frac{1}{(d_u + d_u)},$$

$$PCII(G)(G) = \prod_{uv \in E(G)} \frac{1}{(d_u \cdot d_u)}.$$

Multiplicative atomic bond connectivity index and multiplicative Geometric arithmetic index are defined as

$$ABCII(G) = \prod_{uv \in E(G)} \sqrt{\frac{d_u + d_u - 2}{d_u \cdot d_u}},$$

$$GAII(G) = \prod_{uv \in E(G)} \frac{2\sqrt{d_u \cdot d_u}}{d_u + d_u},$$

$$GA^aII(G) = \prod_{uv \in E(G)} \left(\frac{2\sqrt{d_u \cdot d_u}}{d_u + d_u} \right)^a.$$

In this paper we compute multiplicative indices of circulant graphs.

Definition 1.1. Let n, m , and a_1, a_2, \dots, a_m be positive integers, where $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$ and $a_i \neq a_j$ for all $1 \leq i \leq j \leq m$. An undirected graph with the set of vertices $V = \{v_1, v_2, \dots, v_n\}$ and the set of edges $E = \{v_i v_{i+a_i} : 1 \leq i \leq n, 1 \leq j \leq m\}$ where the indices being taken modulo n , is called the circulant graph, and is denoted by $C_n(a_1, a_2, \dots, a_m)$ The graph of $C_{11}(1, 2, 3)$ is shown in Figure 1.

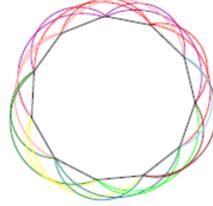


Figure 1. The graph of $C_{11}(1, 2, 3)$.

This is one of the most comprehensive families, as its specializations give some important families. Classes of graphs that are circulant include the Andr’sfai graphs, antiprism graphs, cocktail party graphs, complete graphs, complete bipartite graphs, crown graphs, empty graphs, rook graphs, M’bius ladders, Paley graphs of prime order, prism graphs, and torus grid graphs.

2. Computational Results

In this section, we present our computational results.

Theorem 2.1. *Let $C_n(a_1, a_2, \dots, a_m)$ be the Circulant graph. Then*

- (1) $MZ_1^a(C_n(a_1, a_2, \dots, a_m)) = 2(n - 1)^{an}$,
- (2) $MZ_2^a(C_n(a_1, a_2, \dots, a_m)) = (n - 1)^{2an}$,
- (3) $G^a AII(C_n(a_1, a_2, \dots, a_m)) = 1$.

Proof. Let $C_n(a_1, a_2, \dots, a_m)$, where $n = 3, 4, \dots, n$ and $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$ and $a_i \neq a_j$, when n is odd to be circulant graph. It is clear form the graph of $C_n(a_1, a_2, \dots, a_m)$ that the circulant gaph has only one partition of the vertex set i.e

$$V_1 = \{v \in V(C_n(a_1, a_2, \dots, a_m)) : d_v = n\}.$$

The edge partition of $C_n(a_1, a_2, \dots, a_m)$ has the following partition,

$$E_1 = E_{n-1, n-1} = \{e = uv \in E(C_n(a_1, a_2, \dots, a_m)) : d_u = d_v = n - 1\}.$$

Now,

$$|E_1(C_n(a_1, a_2, \dots, a_m))| = n.$$

(1)

$$\begin{aligned} MZ_1^a(C_n(a_1, a_2, \dots, a_m)) &= \prod_{uv \in E(G)} (d_u + d_v)^a \\ &= (d_u + d_v)^{a|E(C_n(a_1, a_2, \dots, a_m))|} \\ &= ((n - 1) + (n - 1))^{an} \\ &= 2(n - 1)^{an}. \end{aligned}$$

(2)

$$\begin{aligned}
MZ_2^a(C_n(a_1, a_2, \dots, a_m)) &= \prod_{uv \in E(G)} (d_u \times d_v)^a \\
&= (d_u + d_v)^{a|E(C_n(a_1, a_2, \dots, a_m))|} \\
&= ((n-1) \times (n-1))^{an} \\
&= (n-1)^{2an}.
\end{aligned}$$

(3)

$$\begin{aligned}
G^a AII(C_n(a_1, a_2, \dots, a_m)) &= \prod_{uv \in E(C_n(a_1, a_2, \dots, a_m))} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} \right)^a \\
&= \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} \right)^{a|E_1(C_n(a_1, a_2, \dots, a_m))|} \\
&= \left(\frac{2\sqrt{(n-1)^2}}{2(n-1)} \right)^{an} \\
&= 1.
\end{aligned}$$

□

Corollary 2.2. Let $C_n(a_1, a_2, \dots, a_m)$ be the Circulant graph. Then

- (1) $MZ_1(C_n(a_1, a_2, \dots, a_m)) = 2(n-1)^n$,
- (2) $MZ_2(C_n(a_1, a_2, \dots, a_m)) = (n-1)^{2n}$,
- (3) $G AII(C_n(a_1, a_2, \dots, a_m)) = (1)^n = 1$.

Proof. We get our result by putting $\alpha = 1$ in the Theorem 2.1. □

Corollary 2.3. Let $C_n(a_1, a_2, \dots, a_m)$ be the Circulant graph. Then

- (1) $HII_1(C_n(a_1, a_2, \dots, a_m)) = 2(n-1)^{2n}$,
- (2) $HII_2(C_n(a_1, a_2, \dots, a_m)) = (n-1)^{4n}$,

Proof. We get our desired results by putting $\alpha = 2$ in Theorem 2.1. □

Corollary 2.4. Let $C_n(a_1, a_2, \dots, a_m)$ be the Circulant graph. Then

- (1) $XII(C_n(a_1, a_2, \dots, a_m)) = \left(\frac{1}{\sqrt{2(n-1)}} \right)^n$,
- (2) $\chi II(C_n(a_1, a_2, \dots, a_m)) = \left(\frac{1}{n-1} \right)^n$.

Proof. We get our desired results by putting $\alpha = \frac{-1}{2}$ in Theorem 2.1. □

Corollary 2.5. Let $C_n(a_1, a_2, \dots, a_m)$ be the Circulant graph. Then

$$ABCII(C_n(a_1, a_2, \dots, a_m)) = \left(\sqrt{\frac{2(n-2)}{(n-1)^2}} \right)^n.$$

Proof. By using the edge partition given in Theorem 2.1, we have

$$\begin{aligned} ABCII(C_n(a_1, a_2, \dots, a_m)) &= \prod_{uv \in E(C_n(a_1, a_2, \dots, a_m))} \sqrt{\frac{d_u + d_u - 2}{d_u \cdot d_u}} \\ &= \left(\sqrt{\frac{(n-1) + (n-2) - 2}{(n-1)^2}} \right)^n \\ &= \left(\sqrt{\frac{2(n-2)}{(n-1)^2}} \right)^n. \end{aligned}$$

□

Competing Interests

The authors declare that they have no competing interests.

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Ghulam Hussain

Department of Mathematics, The University of Lahore, Pakpattan Campus, Pakpattan 57400, Pakistan.

e-mail: ghulamhussain151ghs@gmail.com

Saba Noreen

Department of Mathematics and Statistics, The University of Lahore, Lahore Pakistan.

e-mail: sabaalvi77@yahoo.com

Waseem Khaild

Department of Mathematics, The University of Lahore, Pakpattan Campus, Pakpattan 57400, Pakistan.

e-mail: waseem.khalid@math.uol.edu.pk