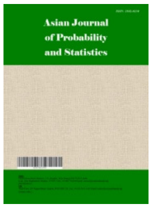

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Class of Tests Based on Subsample Quantiles for Two-sample Scale Problem

Shashank D. Shindhe[1], Sharada V. Bhat[1*] and Parameshwar V. Pandit[2]

[1]*Department of Statistics, Karnatak University, Dharwad, India.*
[2]*Department of Statistics, Bangalore University, Bangalore, India.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

A class of distribution-free tests based on U-statistics with its kernel being function of subsample quantiles is proposed for a two-sample scale problem. The proposed class of tests is a general class of tests that includes numerous members which explore information from the tails of the distributions and tests resistant to outliers. This class of tests includes many existing classes of tests as its subclasses. The distribution of the proposed class of tests is derived and its relevance is discussed. One of its members, which is resistant to outliers in Y-sample is investigated in detail.

Keywords: Two-sample; quantiles; asymptotic relative efficiency (ARE); null distribution; empirical power.

1 Introduction

Suppose $(X_1, X_2, \dots, X_m)$ and $(Y_1, Y_2, \dots, Y_n)$ are two random samples respectively from absolutely continuous distribution functions $F(x)$ and $G(x) = F\left(\frac{x}{\sigma}\right)$, $\sigma > 0$. A two-sample scale problem under consideration is testing $H_0: \sigma = 1$ against $H_1: \sigma > 1$.

**Corresponding author: Email: bhat_sharada@yahoo.com;*

In statistical literature, we find numerous procedures for two-sample scale problem, U-statistics approach being one of the prominent procedures. Construction of distribution-free tests based on U-statistics is of prime focus in nonparametric inference as they are based on symmetric kernels that are functions of order statistics which are sufficient and complete for a given family of continuous distributions. This also implies that the U-statistics are uniformly minimum variance unbiased estimators (UMVUE).

Many distribution-free tests based on U-statistics exist in the literature for two-sample scale problem. Tests due to [1,2,3,4] are some of the earlier tests designed based on the two-sample U-statistics. Also, [5-13] study this problem using U-statistics approach. The asymptotic properties of the U-statistics are discussed in [1]. Some more distribution-free tests proposed for two-sample scale problem that explore [9] are BS_1 due to [14], BS_2 and BS_3 due to [15], BS_4 and BS_5 due to [16], BS_6 due to [17] and BS_7 due to [18]. The test in [14] is based on the subsample medians and hence is resistant to the outliers in both the samples. The tests discussed in [15,16,17] are based on subsample extremes and extract information from the tails of the distribution from which samples are drawn. The test studied in [18] is resistant to outliers in the first sample (X-sample) and is based on the information from the right tail of the second sample (Y-sample). A detailed study of these tests is also contained in [19].

In this paper, we propose a class of tests based on subsample quantiles, $Q_{\alpha,\beta}(b,d)$ for two-sample scale problem. This is a general class of tests that includes [2,3,9,14-18] as its subclasses. Also, it includes the classes of tests which are resistant to outliers in Y-sample and extract information from tails of the distribution of X-sample discussed in this paper. This class of tests is a U-statistic that is developed utilizing the kernel of U-statistics given in [20].

The advantage of the proposed class of tests is that the researcher may use any subclass among the various subclasses available according to the nature of the information contained in the samples under consideration.

The class of tests is proposed in section 2 and its distributional properties are discussed in section 3. In section 4, we discuss about the significance of the class along with one of its members. In section 5, the performance of $Q_{\alpha,\beta}(b,d)$ is studied and conclusions are recorded in section 6.

2 Proposed Class of Tests

In this section, we propose the class of tests $Q_{\alpha,\beta}(b,d)$ for testing $H_0{:}\,\sigma = 1$ against $H_1{:}\,\sigma > 1$. $Q_{\alpha,\beta}(b,d)$ is a two-sample U-statistics based on a symmetric kernel $\phi(\cdot)$ being a function of α^{th} quantile of subsample of size b and β^{th} quantile of subsample of size d respectively drawn from X-sample of size m and Y-sample of size n. The class of tests is defined as

$$Q_{\alpha,\beta}(b,d) = \left(\binom{m}{b}\binom{n}{d}\right)^{-1} \sum_{S} \phi(X_{i_1}, X_{i_2}, \dots, X_{i_b}, Y_{j_1}, Y_{j_2}, \dots, Y_{j_d}) \tag{2.1}$$

where S denotes the sum over all possible $\binom{m}{b}\binom{n}{d}$ arrangements of X and Y sample observations,

$$\phi(x_1, \dots, x_b; y_1, \dots, y_d) = \begin{cases} 1 & \text{if } 0 < \alpha^{th}\text{quantile of}(x_1^+, \dots, x_b^+) < \beta^{th}\text{quantile of}(y_1^+, \dots, y_b^+), x_i, y_j > 0 \\ -1 & \text{if } \beta^{th}\text{quantile of}(y_1^-, \dots, y_b^-) < \alpha^{th}\text{quantile of}(x_1^-, \dots, x_b^-) < 0, x_i, y_j < 0, \\ 0 & \text{Otherwise} \end{cases} \tag{2.2}$$

$i = 1,2, \dots, b$, $j = 1,2, \dots, d$, b and d are positive integers such that $b \le m$ and $d \le n$.

That is,

$$\phi(x_1, x_2, \dots, x_b, y_1, y_2, \dots, y_d) = \begin{cases} 1 & if\ 0 < x_{(r)}^+ < y_{(s)}^+, x_i, y_j > 0 \\ -1 & if\ y_{(s)}^- < x_{(r)}^- < 0, x_i, y_j < 0 \\ 0 & \text{Otherwise} \end{cases} \tag{2.3}$$

where $x_{(r)}^{+}$ and $x_{(r)}^{-}$ are r^{th} order statistics (os) based on a subsample of size b respectively from positive and negative observations of the X-sample. Similarly $y_{(s)}^{+}$ and $y_{(s)}^{-}$ are defined based on Y-sample.

Here, $r = \begin{cases} b\alpha & \text{if } b\alpha \text{ is an integer} \\ [b\alpha]+1 & \text{if } b\alpha \text{ is not an integer} \end{cases}$

$s = \begin{cases} d\beta & if\ d\beta\ is\ an\ integer \\ [d\beta]+1 & if\ d\beta\ is\ not\ an\ integer \end{cases}$

such that $[b\alpha]$ $([d\beta])$ is the largest integer smaller than or equal to $b\alpha$ $(d\beta)$.

For all values of m and n, $2 \le b \le m$ and $2 \le d \le n$, $Q_{\alpha,\beta}(b,d)$ are distribution-free and their large values are significant for testing H_0 against H_1.

$Q_{\alpha,\beta}(b,d)$ can be expressed in terms of ordered ranks alternatively. Following [20] we give the alternative form of $Q_{\alpha,\beta}(b,d)$ as

$$\begin{aligned} Q'_{\alpha,\beta}(b,d) &= \binom{m}{b}\binom{n}{d} Q_{\alpha,\beta}(b,d) \\ &= \sum_{i=1}^{m^+}\sum_{j=1}^{s-1}\binom{i-1}{r-1}\binom{m^+-i}{b-r}\binom{R_{(i)}^{+}-i}{s-j-1}\binom{n^+-R_{(i)}^{+}+i}{d-s+j+1} \\ &\quad -\sum_{j=1}^{n^-}\sum_{i=1}^{r-1}\binom{j-1}{s-1}\binom{n^--j}{d-s}\binom{S_{(j)}^{-}-j}{b-r+i+1}\binom{m^--S_{(j)}^{-}+j}{r-i-1} \end{aligned} \tag{2.4}$$

where $R_{(i)}^{+}\left(S_{(j)}^{+}\right)$ and $R_{(i)}^{-}\left(S_{(j)}^{-}\right)$ are respectively the ranks of $X_{(i)}^{+}\left(Y_{(j)}^{+}\right)$ and $X_{(i)}^{-}\left(Y_{(j)}^{-}\right)$ in the joint rankings of $X_1^+,\ldots,X_{m^+}^+,Y_1^+,\ldots,Y_{n^+}^+$ and $X_1^-,\ldots,X_{m^-}^-,Y_1^-,\ldots,Y_{n^-}^-$ such that $X_{(1)}^{+} < X_{(2)}^{+} < \cdots < X_{(m^+)}^{+}$ are os of positive X-observations, $X_{(1)}^{-} < X_{(2)}^{-} < \cdots < X_{(m^-)}^{-}$ are os of negative X-observations, $Y_{(1)}^{+} < Y_{(2)}^{+} < \cdots < Y_{(n^+)}^{+}$ are os of positive Y-observations, $Y_{(1)}^{-} < Y_{(2)}^{-} < \cdots < Y_{(n^-)}^{-}$ are os of negative Y-observations, $m = m^+ + m^-$ and $n = n^+ + n^-$.

3 Distribution of $Q_{\alpha,\beta}(b,d)$

In this section, we derive the distributional properties viz. mean, null mean and asymptotic variance of the proposed class of tests.

The mean of $Q_{\alpha,\beta}(b,d)$ is given by

$$\begin{aligned} \mu_{\alpha,\beta} &= E\left[Q_{\alpha,\beta}(b,d)\right] \\ &= P\left[0 < X_{(r)}^{+} < Y_{(s)}^{+}\right] - P\left[Y_{(s)}^{-} < X_{(r)}^{-} < 0\right] \\ &= \int_0^\infty \bar{F}_{Y_{(s)}^{+}}(x) f_{X_{(r)}^{+}}(x) f(x)dx - \int_{-\infty}^{0} F_{Y_{(s)}^{-}}(x) f_{X_{(r)}^{-}}(x) f(x)dx \\ &= 1 - \frac{b!}{(r-1)!\,(b-r)!}\sum_{j=s}^{d}\binom{d}{i}\int_0^\infty (2G(x)-1)^j(2\bar{G}(x))^{d-j}(2F(x)-1)^{r-1} \\ &\quad \left(2\bar{F}(x)\right)^{b-r} 2f(x)dx - \frac{b!}{(r-1)!\,(b-r)!}\sum_{j=s}^{d}\binom{d}{i}\int_{-\infty}^{0}(2G(x))^j \\ &\quad \left(1-2G(x)\right)^{d-j}\left(2F(x)\right)^{r-1}\left(1-2F(x)\right)^{b-r} 2f(x)dx. \end{aligned} \tag{3.1}$$

Under $H_0: F(x) = G(x)$, the null mean of $Q_{\alpha,\beta}(b,d)$ is given by

$$\mu_{0_{\alpha,\beta}} = E_{H_0}\left[Q_{\alpha,\beta}(b,d)\right] = 1 - \frac{2b!}{(r-1)!\,(b-r)!}\sum_{j=s}^{d}\binom{d}{j}B(j+r, b+d-j-r+1) \tag{3.2}$$

where $B(p,q) = \dfrac{(p-1)!\,(q-1)!}{(p+q-1)!}$.

According to [1], $Q_{\alpha,\beta}(b,d)$ has asymptotic normal distribution with mean $\mu_{0_{\alpha,\beta}}$ and variance $\sigma^2_{\alpha,\beta}$ as its limiting distribution with $m+n=N\to\infty$ such that $0<\lambda=\lim_{N\to\infty}\frac{m}{N}<1$. The mean of $Q_{\alpha,\beta}(b,d)$ is $\mu_{0_{\alpha,\beta}}$ and the asymptotic variance $\sigma^2_{\alpha,\beta}$ is given by

$$\sigma^2_{\alpha,\beta} = \frac{b^2\xi_{10}}{\lambda} + \frac{d^2\xi_{01}}{1-\lambda}. \tag{3.3}$$

Here, ξ_{10} is given by

$$\begin{aligned}
\xi_{10} &= Cov[\phi(X_1,\dots,X_b;Y_1,\dots,Y_d),\phi(X_1,X_{b+1},\dots,X_{2b-1};Y_{d+1},\dots,Y_{2d})]\\
&= \int_{-\infty}^{\infty} P^2[(0<r^{th}\text{ os }(x,X_2,\dots,X_b)<s^{th}\text{ os }(Y_1,\dots,Y_d))\\
&\quad -(s^{th}\text{ os }(Y_1,\dots,Y_d)<r^{th}\text{ os }(x,X_2,\dots,X_b)<0)]2f(x)dx-\left(\mu_{0_{\alpha,\beta}}\right)^2\\
&= \int_0^{\infty} P_1^2 f(x)dx + \int_{-\infty}^{0} P_2^2 f(x)dx - 2\int_0^{\infty} P_1 f(x)dx\int_{-\infty}^{0} P_2 f(x)dx - \left(\mu_{0_{\alpha,\beta}}\right)^2,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
P_1 &= P\left[0<r^{th}\text{ os }(x,X_2,\dots,X_b)<Y_{(s)}\right]\\
&= P\left[0<x<Y_{(s)};X_2<\cdots<X_{r-1}<x<\cdots<X_{b-1}\right]\\
&\quad +P\left[0<X_{r-1}<Y_{(s)};x<X_2<\cdots<X_{b-1}\right]\\
&\quad +P\left[0<X_r<Y_{(s)};X_2<\cdots<X_r<\cdots<x\right]\\
&= P_{11}+P_{12}+P_{13},
\end{aligned} \tag{3.5}$$

$$P_{11} = \sum_{i=r-1}^{b-1}\binom{b-1}{i}(2F(x)-1)^i(2\bar{F}(x))^{b-1-i}\left[1-\sum_{j=s}^{d}\binom{d}{j}(2F(x)-1)^j(2\bar{F}(x))^{d-j}\right],$$

$$\begin{aligned}
P_{12} = (r-1)\binom{b-1}{r-1}\Bigg[&B(b-r+1,r-1)\sum_{i=b-r+1}^{b-1}\binom{b-1}{i}(2F(x)-1)^{b-1-i}\left(2\bar{F}(x)\right)^i\\
&-\sum_{j=s}^{d}\binom{d}{j}B(b+d-j-r+1,j+r\\
&-1)\sum_{k=b+d-j-r+1}^{b+d-1}\binom{b+d-1}{k}(2F(x)-1)^{b+d-1-k}\left(2\bar{F}(x)\right)^k\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
P_{13} &= (b-r)\binom{b-1}{r-1}\Bigg[B(r,b-r)\sum_{i=r}^{b-1}\binom{b-1}{i}(2F(x)-1)^i(2\bar{F}(x))^{b-1-i}\\
&\quad -\sum_{j=s}^{d}\binom{d}{j}B(j+r,b+d-j-r)\sum_{k=j+r}^{b+d-1}\binom{b+d-1}{k}(2F(x)-1)^k(2\bar{F}(x))^{b+d-1-k}\Bigg].
\end{aligned}$$

Similarly,

$$\begin{aligned} P_2 &= P\left[Y_{(s)} < r^{th} \text{ os } (x, X_2, \ldots, X_b) < 0\right] \\ &= P\left[Y_{(s)} < x < 0; X_2 < \cdots < X_{r-1} < x < \cdots < X_{b-1}\right] \\ &\quad + P\left[Y_{(s)} < X_{r-1} < 0; x < X_2 < \cdots < X_{b-1}\right] \\ &\quad + P\left[Y_{(s)} < X_r < 0; X_2 < \cdots < X_r < \cdots < x\right]. \end{aligned} \tag{3.6}$$

Also,

$$\begin{aligned} \xi_{01} &= Cov[\phi(X_1, \ldots, X_b; Y_1, \ldots, Y_d), \phi(X_{b+1}, \ldots, X_{2b}; Y_1, Y_{d+1}, \ldots, Y_{2d-1})] \\ &= \int_{-\infty}^{\infty} P^2[(0 < r^{th} \text{ os}(X_1, \ldots, X_b) < s^{th} \text{ os } (y, Y_2, \ldots, Y_d)) \\ &\quad -(s^{th} \text{ os } (y, Y_2, \ldots, Y_d) < r^{th} \text{ os } (X_1, \ldots, X_b) < 0)]2f(y)dy - \left(\mu_{0_{\alpha,\beta}}\right)^2 \\ &= \int_0^{\infty} (P_1')^2 f(y)dy + \int_{-\infty}^{0} (P_2')^2 f(y)dy - 2\int_0^{\infty} P_1' f(y)dy \int_{-\infty}^{0} P_2' f(y)dy - \left(\mu_{0_{\alpha,\beta}}\right)^2, \end{aligned} \tag{3.7}$$

where,

$$\begin{aligned} P_1' &= P\left[0 < X_{(r)} < s^{th} \text{ os } (y, Y_2, \ldots, Y_d)\right] \\ &= P\left[0 < X_{(r)} < y; Y_2 < \cdots < Y_{s-1} < y < \cdots < Y_{d-1}\right] \\ &\quad + P\left[0 < X_{(r)} < Y_{s-1}; y < Y_2 < \cdots < Y_{d-1}\right] \\ &\quad + P\left[0 < X_{(r)} < Y_s; Y_2 < \cdots < Y_s < \cdots y\right], \end{aligned}$$

and

$$\begin{aligned} P_2' &= P\left[s^{th} \text{ os } (y, Y_2, \ldots, Y_d) < X_{(r)} < 0\right] \\ &= P\left[y < X_{(r)} < 0; Y_2 < \cdots < Y_{s-1} < y < \cdots < Y_{d-1}\right] \\ &\quad + P\left[Y_{s-1} < X_{(r)} < 0; y < Y_2 \ldots < Y_{d-1}\right] \\ &\quad + P\left[Y_s < X_{(r)} < 0; Y_2 < \cdots < Y_s < \cdots < y\right]. \end{aligned}$$

Since the kernel under consideration is symmetric, from (3.4) and (3.7) we have

$$b^2\xi_{10} = d^2\xi_{01}. \tag{3.8}$$

4 Importance of $Q_{\alpha,\beta}(b, d)$

$Q_{\alpha,\beta}(b, d)$ is a general class of two-sample scale tests that includes many classes of tests which are existing in literature [2,3,9,14-18]. For different values of b, d, α (or r) and β (or s) we get various subclasses of $Q_{\alpha,\beta}(b, d)$. For $b = d, r = s = \frac{b+1}{2}$, we get test due to [9]. For $b \neq d, r = \frac{b+1}{2}$ and $s = \frac{d+1}{2}$ where b and d are odd positive integers, we get the class of tests BS_1 which is outlier resistant to $\frac{b-1}{2}$ outliers in X-sample and $\frac{d-1}{2}$ outliers in the Y-sample. For $r = b$, $s = d$ and $r = 1$, $s = 1$, we get the classes of tests BS_2, BS_3, BS_4, BS_5 and BS_6 which explore the information in the tails of the probability distributions. For $r = \frac{b+1}{2}$, b is an odd positive integer and $s = d$ we get the class of tests BS_7 which is outlier resistant to $\frac{b-1}{2}$ outliers in X-sample and is based on extreme observations from Y-sample.

Sometimes the outliers present in Y-sample contribute to variation in the variance of $G(x)$ leading to faulty analysis. Hence a class of tests which is outlier resistant in Y-sample becomes a necessity. In this paper, we carry out detailed discussion on such a class of tests, $Q^*(b, d)$ which is obtained from $Q_{\alpha,\beta}(b, d)$ by taking $\phi(\cdot)$ to be the function of maximum os and the median of subsamples of sizes b and d respectively from X and Y

samples. $Q^*(b, d)$ emerges from $Q_{\alpha,\beta}(b, d)$ by substituting $r = b$ and $s = \frac{d+1}{2}$ (d being an odd positive integer) in (2.3). This subclass of tests explores information from right tail of X-sample and is resistant to $\frac{d-1}{2}$ outliers in Y-sample. The alternative form of $Q^*(b, d)$ in terms of ordered ranks and its distributional properties are respectively obtained by substituting for r and s in (2.4) and (3.2), (3.3). Also, an equivalent statistic to $Q^*(b, d)$ exists when $r = 1$ and $s = \frac{d+1}{2}$ is substituted in (2.3).

Using the alternative expression of $Q^*(b, d)$ in terms of ordered ranks, its null distribution is obtained and is presented in Fig. 1. The null distribution is generated using 10000 random samples from Uniform distribution.

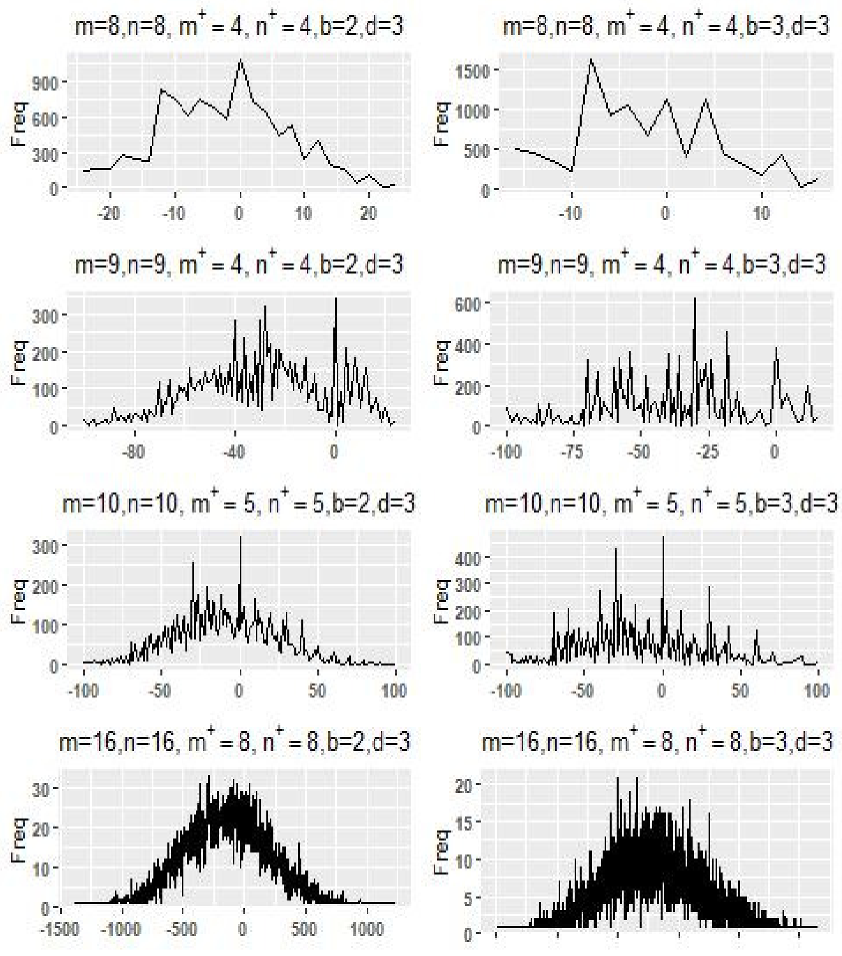

Fig. 1. Null distribution of $Q'(b, d)$ for different values of m, n, m^+, n^+, b and d

It is observed that, for larger values of m, n and smaller values of b, d the null distribution of $Q^*(b, d)$ follows Normal distribution.

The large sample performance of the proposed class of tests can be compared with any other test using Pitman ARE given in [21]. The ARE of T_1 with respect to (wrt) any test T_2 is given by

$$ARE(T_1, T_2) = \frac{e(T_1)}{e(T_2)}, \tag{4.1}$$

where $e(T_i), i = 1,2$ is known as the efficacy of the test T_i and is defined as

$$e(T_i) = \left[\frac{\frac{d}{d\sigma}\left[E_{H_1}(T_i)\right]_{\sigma=1}}{\sigma}\right]^2.$$

The efficacy of $Q_{\alpha,\beta}(b,d)$ is given by

$$e\left[Q_{\alpha,\beta}(b,d)\right] = \frac{\left\{bd\binom{b-1}{r-1}\binom{d-1}{s-1}[I_1 - I_2]\right\}^2}{\sigma^2_{\alpha,\beta}}, \tag{4.2}$$

where $I_1 = \int_0^\infty x(2F(x)-1)^{r+s-2}(2\bar{F}(x))^{b+d-r-s}(2f(x))^2 dx$,

$I_2 = \int_{-\infty}^0 x(2F(x))^{r+s-2}(1-2F(x))^{b+d-r-s}(2f(x))^2 dx$ and $\sigma^2_{\alpha,\beta}$ is given by (3.3).

5 Performance of $Q_{\alpha,\beta}(b,d)$

In this section, we study the large sample and small sample performances of $Q_{\alpha,\beta}(b,d)$. The efficiencies of different subclasses of tests of $Q_{\alpha,\beta}(b,d)$ are already discussed in literature and here we discuss about the performance of the subclass of tests, namely $Q^*(b,d)$. We compute its efficacy and compare its performance with its competitors. The efficacies of $Q^*(b,d)$ are presented in Table 1 of appendix, in Figs. 2 and 3.

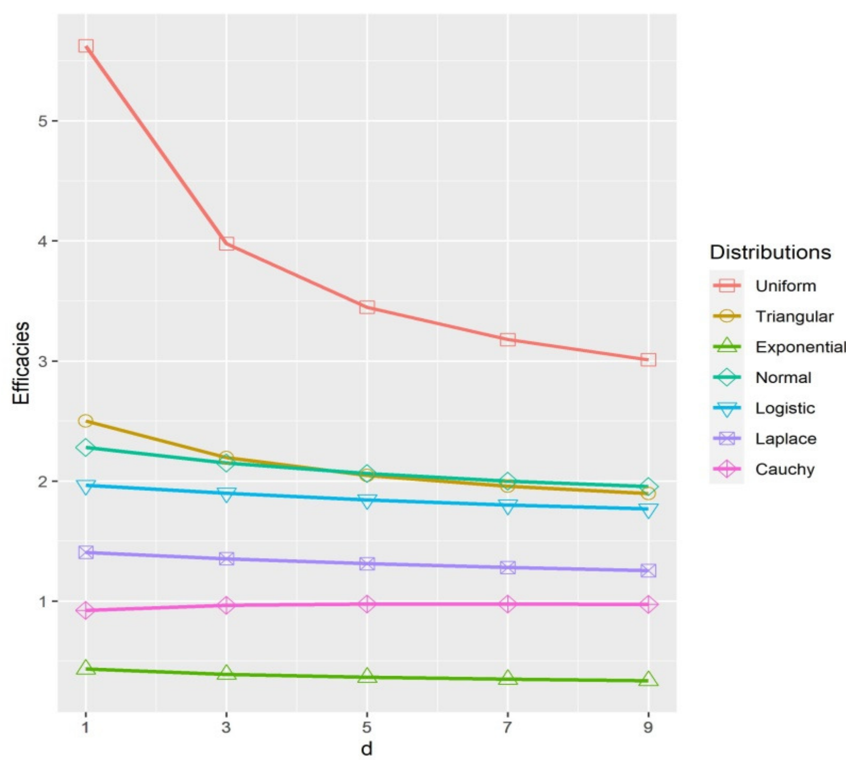

Fig. 2. Efficacy of $Q^*(b,d)$ for a given b and increasing values of d

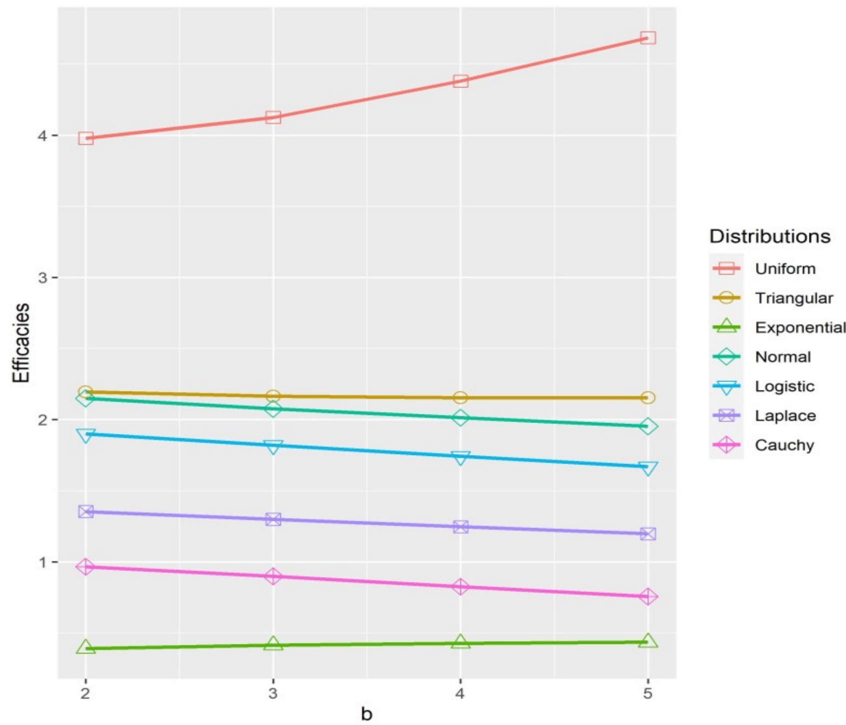

Fig. 3. Efficacy of $Q^*(b, d)$ for a given d and increasing values of b

We observe that, the efficacy of $Q^*(b, d)$ is high for smaller values of d for a given $b(< 5)$ and it decreases as d increases for all the distributions under consideration, Cauchy distribution being an exception. For $b = 5$ and $d < 9$ under Logistic and Laplace distributions the efficacy increases with increasing value of d, whereas, for a given d and increasing b the efficacy values are decreasing except for Uniform and Exponential distributions.

The large sample performance of $Q^*(b, d)$ is compared with various subclasses of tests in terms of Pitman ARE. We take various subclasses of tests of $Q_{\alpha,\beta}(b, d)$ as its competitors. The ARE of $Q^*(b, d)$ wrt BS_1, BS_2, BS_4 and BS_5 are respectively given in Tables 2, 3 and 4 and the $ARE(Q^*(b, d), BS_7)$ is given in Table 5 of appendix.

The $ARE(Q^*(b, d), BS_3) = ARE(Q^*(b, d), BS_2)$ since $ARE(BS_2, BS_3) = 1$.

And $ARE(Q^*(b, d), BS_6) = 0.5 * ARE(Q^*(b, d), BS_2)$ since $ARE(BS_6, BS_2) = 2$. In all the tables, t is taken as the sum of subsamples of the competitors.

Table 2 reveals that $Q^*(b, d)$ outperforms BS_1 under Uniform, Triangular, Exponential, Normal, Logistic and Laplace distributions when smaller values of d are chosen for a given b. It is also observed that, under all these distributions, the ARE decreases with increasing values of d.

Table 3 shows that, $Q^*(b, d)$ is better than BS_2 under Uniform, Logistic and Laplace distributions. The ARE values for a given set of b and d decrease with increasing values of t. That is $Q^*(b, d)$ outperforms BS_2 when the values of b, d and t are smaller.

Table 4 reveals that, $Q^*(b, d)$ outperforms BS_4 under Cauchy distribution. The ARE values of $Q^*(b, d)$ wrt to BS_4 are increasing with increasing values of b, d and t. $Q^*(b, d)$ outperforms BS_5 under Uniform, Triangular, Exponential, Logistic, Laplace and Cauchy distributions for all values of b, d and t considered. Also, the ARE values in these cases are found to be increasing with increasing values of b, d and t.

According to Table 5, $Q^*(b,d)$ is equivalent to BS_7 when $b = d$. In case of light tailed distributions like Uniform, Triangular and Exponential distributions, $Q^*(b,d)$ is better than BS_7 when $b > d$. In case of Normal, Logistic, Laplace and Cauchy distributions $Q^*(b,d)$ is better than BS_7 when $b < d$.

The small sample performance of $Q^*(b,d)$ is studied in terms of its empirical power and is given in Table 6 of appendix. The empirical power of the class of tests is obtained from Monte-Carlo simulation technique using 10000 random samples from a specified distribution under different alternative hypotheses. The alternative hypothesis considered for the study included different magnitudes of shifts in the scale parameter σ. According to Table 6, the empirical power is high in case of uniform distribution and decreases for distributions with thicker tails. The empirical power is found to increase with increasing values of m, n and b, d. Also, for most of the combinations of sample and subsample sizes considered, the highest empirical power is achieved when smallest values of subsamples, i,e. $b = 2$ and $d = 3$ are chosen.

6 Conclusions

Based on our study, we arrive at the following conclusions.

1. The proposed class of tests $Q_{\alpha,\beta}(b,d)$ comprises of numerous subclasses of tests which can be chosen as per the information available.
2. The member BS_1 is useful in cases where there are outliers present in both samples.
3. In case of information available on extreme order observations, the tests BS_2, BS_3, BS_4, BS_5 and BS_6 are employable.
4. The class of tests BS_7 can be used when an outlier is present in X-sample whereas the class of tests $Q^*(b,d)$ can be made use of when the outliers are present in the Y-sample.
5. $Q^*(b,d)$ is found to be better than BS_1 under all distributions considered for smaller values of b and d, Cauchy being an exception and is better than BS_2, BS_3, BS_4 and BS_5 under light and heavy tailed distributions.
6. $Q^*(b,d)$ performs better than BS_6 for heavy tailed distributions.
7. The performance of $Q^*(b,d)$ is equivalent to BS_7 for $b = d$, is better than BS_7 for light tailed distributions when $b > d$ and heavy tailed distributions when $b < d$.
8. The empirical power of $Q^*(b,d)$ is better under light tailed distributions as compared to distributions with heavier tails.
9. The general class of tests $Q_{\alpha,\beta}(b,d)$ is highly useful for two-sample scale problem as it contains umpteen subclasses and their members which are applicable in multitude of scenarios.
10. The proposed class of tests may be modified by taking any common quantile other than median. Also, it may be extended to test two-sided alternatives and for k-sample problem for ordered and umbrella alternatives.

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Competing Interests

Authors have declared that no competing interests exist.

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Appendix

Table 1. Efficacy of $Q^*(b,d)$ for different values of b and d

b	d	Uniform	Triangular	Exponential	Normal	Logistic	Laplace	Cauchy
2	1	5.6250	2.5000	0.4340	2.2794	1.9635	1.4061	0.9238
	3	3.9773	2.1948	0.3905	2.1495	1.8991	1.3535	0.9659
	5	3.4483	2.0475	0.3654	2.0620	1.8437	1.3113	0.9753
	7	3.1780	1.9572	0.3490	2.0002	1.8008	1.2794	0.9756
	9	3.0103	1.8949	0.3374	1.9544	1.7676	1.2548	0.9730
3	1	6.2222	2.4584	0.4564	2.1467	1.8198	1.3070	0.8134
	3	4.1250	2.1645	0.4136	2.0770	1.8191	1.2985	0.8984
	5	3.5065	2.0249	0.3858	2.0160	1.7928	1.2763	0.9306
	7	3.2055	1.9396	0.3668	1.9681	1.7654	1.2549	0.9438
	9	3.0243	1.8794	0.3528	1.9296	1.7405	1.2361	0.9487
4	1	7.0313	2.4517	0.4626	2.0742	1.6934	1.2207	0.7101
	3	4.3794	2.1539	0.4271	2.0127	1.7420	1.2460	0.8258
	5	3.6296	2.0144	0.3997	1.9716	1.7393	1.2398	0.8775
	7	3.2766	1.9302	0.3801	1.9364	1.7270	1.2288	0.9043
	9	3.0693	1.8723	0.3646	1.9052	1.7110	1.2160	0.9179
5	1	7.9200	2.4567	0.4625	1.9425	1.5854	1.1469	0.6234
	3	4.6847	2.1526	0.4343	1.9533	1.6689	1.1965	0.7563
	5	3.7884	2.0108	0.4094	1.9313	1.6881	1.2053	0.8241
	7	3.3741	1.9259	0.3902	1.9062	1.6884	1.2027	0.8621
	9	3.1351	1.8671	0.3742	1.8812	1.6805	1.1954	0.8840

Table 2. ARE of $Q^*(b,d)$ wrt BS_1

b	d	t	Uniform	Triangular	Exponential	Normal	Logistic	Laplace	Cauchy
2	3	6	1.1278	1.0481	1.1648	1.0193	1.0072	1.0092	0.9683
		8	1.2315	1.1035	1.2013	1.0573	1.0376	1.0409	0.9741
		10	1.3051	1.1441	1.2297	1.0864	1.0613	1.0655	0.9806
	5	6	0.9778	0.9778	1.0897	0.9778	0.9778	0.9778	0.9778
		8	1.0677	1.0294	1.1239	1.0143	1.0073	1.0085	0.9836
		10	1.1315	1.0674	1.1505	1.0421	1.0303	1.0323	0.9902
	7	6	0.9011	0.9347	1.0408	0.9485	0.9551	0.9540	0.9780
		8	0.9840	0.9840	1.0735	0.9839	0.9839	0.9839	0.9839
		10	1.0428	1.0203	1.0988	1.0109	1.0064	1.0071	0.9905
3	3	6	1.1696	1.0337	1.2335	0.9849	0.9647	0.9682	0.9006
		8	1.2772	1.0883	1.2722	1.0217	0.9939	0.9986	0.9060
		10	1.3536	1.1284	1.3023	1.0497	1.0166	1.0222	0.9121
	5	6	0.9943	0.9670	1.1508	0.9560	0.9508	0.9517	0.9329
		8	1.0857	1.0180	1.1869	0.9917	0.9795	0.9815	0.9385
		10	1.1506	1.0556	1.2149	1.0189	1.0019	1.0047	0.9448
	7	6	0.9089	0.9263	1.0941	0.9332	0.9362	0.9357	0.9461
		8	0.9925	0.9752	1.1284	0.9681	0.9645	0.9651	0.9518
		10	1.0519	1.0111	1.1551	0.9947	0.9866	0.9879	0.9582
4	3	6	1.2418	1.0286	1.2738	0.9544	0.9238	0.9291	0.8278
		8	1.3560	1.0829	1.3138	0.9901	0.9517	0.9583	0.8328
		10	1.4371	1.1228	1.3448	1.0173	0.9735	0.9809	0.8384
	5	6	1.0292	0.9620	1.1921	0.9349	0.9224	0.9245	0.8797
		8	1.1238	1.0128	1.2295	0.9698	0.9503	0.9535	0.8850
		10	1.1910	1.0501	1.2585	0.9965	0.9720	0.9760	0.8909
	7	6	0.9291	0.9218	1.1338	0.9182	0.9159	0.9163	0.9065
		8	1.0145	0.9704	1.1694	0.9525	0.9436	0.9450	0.9120
		10	1.0752	1.0062	1.1970	0.9787	0.9651	0.9673	0.9181

b	d	t	Uniform	Triangular	Exponential	Normal	Logistic	Laplace	Cauchy
5	3	6	1.3283	1.0279	1.2953	0.9262	0.8851	0.8922	0.7582
		8	1.4505	1.0822	1.3360	0.9608	0.9118	0.9202	0.7628
		10	1.5372	1.1221	1.3675	0.9872	0.9327	0.9419	0.7679
	5	6	1.0742	0.9602	1.2210	0.9158	0.8953	0.8987	0.8262
		8	1.1730	1.0110	1.2594	0.9500	0.9223	0.9269	0.8311
		10	1.2431	1.0482	1.2891	0.9761	0.9434	0.9488	0.8367
	7	6	0.9567	0.9197	1.1637	0.9039	0.8954	0.8968	0.8643
		8	1.0447	0.9683	1.2002	0.9377	0.9225	0.9249	0.8695
		10	1.1072	1.0040	1.2286	0.9634	0.9436	0.9468	0.8753

Table 3. ARE of $Q^*(b,d)$ wrt BS_2

b	d	t	Uniform	Logistic	Laplace
2	3	5	1.5712	15.6304	20.6017
		6	1.1299	12.3881	16.1712
		7	0.8811	10.7051	13.8540
	5	5	1.3623	15.1742	19.9589
		6	0.9796	12.0265	15.6667
		7	0.7639	10.3927	13.4217
	7	5	1.2555	14.8218	19.4730
		6	0.9029	11.7472	15.2853
		7	0.7041	10.1513	13.0950
3	3	5	1.6296	14.9721	19.7640
		6	1.1719	11.8663	15.5137
		7	0.9138	10.2543	13.2907
	5	5	1.3853	14.7557	19.4261
		6	0.9962	11.6948	15.2485
		7	0.7768	10.1061	13.0634
	7	5	1.2664	14.5299	19.1010
		6	0.9107	11.5158	14.9932
		7	0.7102	9.9514	12.8448
4	3	5	1.7301	14.3372	18.9654
		6	1.2441	11.3632	14.8868
		7	0.9702	9.8195	12.7536
	5	5	1.4339	14.3149	18.8711
		6	1.0311	11.3455	14.8128
		7	0.8041	9.8042	12.6902
	7	5	1.2944	14.2143	18.7035
		6	0.9308	11.2657	14.6813
		7	0.7259	9.7353	12.5775
5	3	5	1.8507	13.7360	18.2116
		6	1.3309	10.8866	14.2951
		7	1.0378	9.4077	12.2467
	5	5	1.4966	13.8941	18.3448
		6	1.0763	11.0120	14.3997
		7	0.8393	9.5160	12.3363
	7	5	1.3329	13.8963	18.3056
		6	0.9585	11.0137	14.3689
		7	0.7475	9.5175	12.3099

Table 4. ARE of $Q^{*}(b, d)$ wrt BS_4 and BS_5

b	d	t	BS_4	BS_5						
			Cauchy	Uniform	Triangular	Exponential	Normal	Logistic	Laplace	Cauchy
2	3	5	1.3604	3.5354	2.4688	2.1696	0.1420	2.0915	1.8799	1.3604
		6	1.5504	4.5196	3.0190	2.5524	0.1731	2.5220	2.2153	1.5504
		7	1.7498	5.5087	3.5688	2.9363	0.2046	2.9627	2.5490	1.7498
	5	5	1.3737	3.0652	2.3032	2.0298	0.1363	2.0305	1.8213	1.3737
		6	1.5655	3.9186	2.8164	2.3880	0.1660	2.4484	2.1462	1.5655
		7	1.7669	4.7761	3.3293	2.7471	0.1963	2.8762	2.4695	1.7669
	7	5	1.3741	2.8249	2.2016	1.9387	0.1322	1.9833	1.7769	1.3741
		6	1.5660	3.6114	2.6922	2.2809	0.1611	2.3916	2.0939	1.5660
		7	1.7674	4.4017	3.1825	2.6239	0.1904	2.8094	2.4094	1.7674
3	3	5	1.2653	3.6667	2.4348	2.2976	0.1372	2.0034	1.8035	1.2653
		6	1.4420	4.6875	2.9774	2.7031	0.1672	2.4158	2.1252	1.4420
		7	1.6275	5.7133	3.5196	3.1095	0.1977	2.8379	2.4454	1.6275
	5	5	1.3107	3.1169	2.2777	2.1435	0.1332	1.9745	1.7726	1.3107
		6	1.4937	3.9847	2.7852	2.5218	0.1623	2.3809	2.0889	1.4937
		7	1.6858	4.8567	3.2925	2.9010	0.1919	2.7969	2.4036	1.6858
	7	5	1.3293	2.8494	2.1818	2.0379	0.1301	1.9443	1.7430	1.3293
		6	1.5149	3.6427	2.6680	2.3976	0.1585	2.3445	2.0539	1.5149
		7	1.7098	4.4398	3.1539	2.7581	0.1873	2.7541	2.3633	1.7098
4	3	5	1.1631	3.8928	2.4228	2.3727	0.1330	1.9185	1.7306	1.1631
		6	1.3255	4.9766	2.9627	2.7914	0.1621	2.3134	2.0393	1.3255
		7	1.4960	6.0656	3.5022	3.2112	0.1916	2.7176	2.3466	1.4960
	5	5	1.2359	3.2263	2.2660	2.2205	0.1303	1.9155	1.7220	1.2359
		6	1.4085	4.1245	2.7709	2.6123	0.1588	2.3098	2.0292	1.4085
		7	1.5897	5.0271	3.2755	3.0051	0.1877	2.7134	2.3349	1.5897
	7	5	1.2736	2.9125	2.1712	2.1119	0.1280	1.9020	1.7067	1.2736
		6	1.4515	3.7234	2.6551	2.4846	0.1559	2.2935	2.0112	1.4515
		7	1.6382	4.5382	3.1386	2.8582	0.1843	2.6943	2.3142	1.6382
5	3	5	1.0653	4.1642	2.4213	2.4128	0.1291	1.8380	1.6618	1.0653
		6	1.2140	5.3235	2.9609	2.8386	0.1573	2.2164	1.9583	1.2140
		7	1.3702	6.4885	3.5001	3.2654	0.1859	2.6036	2.2533	1.3702
	5	5	1.1607	3.3675	2.2619	2.2744	0.1276	1.8592	1.6740	1.1607
		6	1.3228	4.3050	2.7659	2.6758	0.1555	2.2419	1.9726	1.3228
		7	1.4930	5.2471	3.2696	3.0782	0.1838	2.6336	2.2698	1.4930
	7	5	1.2143	2.9992	2.1663	2.1676	0.1260	1.8595	1.6704	1.2143
		6	1.3839	3.8342	2.6491	2.5501	0.1535	2.2422	1.9684	1.3839
		7	1.5619	4.6733	3.1315	2.9336	0.1815	2.6340	2.2649	1.5619

Table 5. ARE of $Q^{*}(b, d)$ wrt BS_7

b	d	Uniform	Triangular	Exponential	Normal	Logistic	Laplace	Cauchy
3	3	1.0000	1.0002	0.9999	1.0000	1.0000	1.0000	1.0000
3	5	0.7485	0.9406	0.8884	1.0321	1.0743	1.0667	1.2304
3	7	0.5970	0.8974	0.8366	1.0641	1.1460	1.1310	1.4791
5	3	1.3360	1.0629	1.1257	0.9689	0.9309	0.9375	0.8127
5	5	1.0000	0.9998	1.0002	1.0000	1.0000	1.0000	1.0000

Table 6. Empirical power of $Q'(b,d)$ for different values of m, n, m^+, n^+, b, d and various distributions for 10% level of significance

m	n	m^+	n^+	b	d	Distribution	σ						
							1.2	**1.5**	**2**	**2.5**	**3**	**4**	**5**
8	8	4	4	2	3	Uniform	0.1187	0.1661	0.1870	0.2133	0.2242	0.2509	0.2594
						Normal	0.1407	0.1606	0.1974	0.2175	0.2386	0.2586	0.2708
						Logistic	0.1340	0.1541	0.1792	0.2114	0.2118	0.2398	0.2628
						Laplace	0.1277	0.1436	0.1703	0.1831	0.2019	0.2187	0.2251
						Cauchy	0.1210	0.1309	0.1414	0.1451	0.1476	0.1617	0.1710
8	8	4	4	3	3	Uniform	0.1488	0.1928	0.2288	0.2349	0.2529	0.2607	0.2675
						Normal	0.1688	0.1925	0.2179	0.2309	0.2480	0.2483	0.2396
						Logistic	0.1605	0.1798	0.2004	0.2121	0.2293	0.2316	0.2312
						Laplace	0.1619	0.1762	0.1884	0.2075	0.2084	0.2195	0.2256
						Cauchy	0.1573	0.1499	0.1596	0.1641	0.1603	0.1799	0.1674
10	10	5	5	2	3	Uniform	0.1055	0.1475	0.1750	0.2043	0.2309	0.2493	0.2693
						Normal	0.1179	0.1478	0.1797	0.2128	0.2341	0.2481	0.2538
						Logistic	0.1173	0.1381	0.1683	0.1962	0.2019	0.2379	0.2377
						Laplace	0.1158	0.1291	0.1510	0.1683	0.1920	0.2043	0.2183
						Cauchy	0.1051	0.1141	0.1180	0.1362	0.1348	0.1455	0.1537
10	10	5	5	3	3	Uniform	0.1160	0.1483	0.1908	0.2089	0.2247	0.2444	0.2519
						Normal	0.1250	0.1537	0.1837	0.1995	0.2149	0.2212	0.2210
						Logistic	0.1271	0.1443	0.1677	0.1881	0.1956	0.2077	0.2080
						Laplace	0.1162	0.1356	0.1576	0.1708	0.1801	0.1925	0.2009
						Cauchy	0.1128	0.1173	0.1233	0.1197	0.1270	0.1353	0.1354
14	14	7	7	2	3	Uniform	0.0605	0.0937	0.1262	0.1442	0.1620	0.1922	0.2081
						Normal	0.0771	0.0925	0.1299	0.1439	0.1675	0.1806	0.1803
						Logistic	0.0753	0.0873	0.1172	0.1380	0.1443	0.1590	0.1771
						Laplace	0.0676	0.0832	0.1011	0.1098	0.1223	0.1449	0.1554
						Cauchy	0.0650	0.0680	0.0772	0.0729	0.0778	0.0888	0.0925
14	14	7	7	2	5	Uniform	0.0977	0.1473	0.1916	0.2199	0.2411	0.2728	0.2926
						Normal	0.1265	0.1529	0.1960	0.2219	0.2412	0.2761	0.2829
						Logistic	0.1123	0.1480	0.1739	0.2026	0.2300	0.2440	0.2606
						Laplace	0.1101	0.1324	0.1526	0.1852	0.1834	0.2141	0.2204
						Cauchy	0.1060	0.1079	0.1124	0.1300	0.1260	0.1372	0.1509
30	30	15	15	2	3	Uniform	0.1002	0.1952	0.2716	0.3465	0.3806	0.4518	0.4894
						Normal	0.1383	0.2001	0.2768	0.3433	0.3794	0.4469	0.4623
						Logistic	0.1342	0.1864	0.2558	0.3005	0.3446	0.4018	0.4200
						Laplace	0.1268	0.1544	0.2122	0.2487	0.2810	0.3207	0.3530
						Cauchy	0.1095	0.1214	0.1360	0.1483	0.1568	0.1751	0.1781
50	50	25	25	2	3	Uniform	0.1039	0.2226	0.3336	0.4251	0.4988	0.5931	0.6407
						Normal	0.1466	0.2319	0.3489	0.4456	0.5040	0.5690	0.6028
						Logistic	0.1386	0.2039	0.3055	0.3826	0.4426	0.5067	0.5365
						Laplace	0.1304	0.1778	0.2489	0.3036	0.3522	0.4115	0.4604
						Cauchy	0.1060	0.1189	0.1426	0.1594	0.1709	0.1885	0.2026
100	100	50	50	2	3	Uniform	0.0967	0.2991	0.4789	0.6147	0.6987	0.7958	0.8473
						Normal	0.1789	0.3024	0.4942	0.6262	0.7111	0.7895	0.8241
						Logistic	0.1622	0.2687	0.4257	0.5479	0.6249	0.7216	0.7733
						Laplace	0.1387	0.2122	0.3260	0.4182	0.4878	0.5926	0.6541
						Cauchy	0.1085	0.1257	0.1560	0.1819	0.2005	0.2326	0.2635

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